1. Prove that for any $p$ in $X$, $\text{farthest}(p)$ is in $\text{CH}(X)$.

**Solution:** Assume for the sake of contradiction that there exists a point $p$ in $X$ such that $q = \text{farthest}(p)$ is in $X \setminus \text{CH}(X)$. Now consider the line segment $s$ with endpoints $p$ and $q$. By the definition of convex hull and the fact that $q$ is not in $\text{CH}(X)$, this line segment lies strictly in the interior of $\text{CH}(X)$. Therefore, we may extend this segment beyond $q$ until it hits an edge, say $e = (b,c)$ of the convex hull, where $b$ and $c$ are the vertices of $\text{CH}(X)$ adjacent to $e$. Call this extension of $s$ to be $s'$, and call the intersection point of $s'$ and $e$ to be $d$. Now because we assumed that $q$ is the farthest point from $p$, then $q$ must be farther from $p$ than both $b$ and $c$ are. Moreover, as the label of $b$ or $c$ does not matter at this point, let $\text{distance}(p,c) > \text{distance}(p,b)$, meaning that $c$ is the farther of the pair $b$ and $c$ from $p$. This means that

$$\text{distance}(p,q) > \text{distance}(p,c) > \text{distance}(p,b).$$

Now, consider the circle $C$ centered at $p$ with radius $r = \text{distance}(p,c)$. By the distances given, $b$ must lie strictly in the interior of $C$, $c$ must lie exactly on the boundary of $C$, and $q$ must lie strictly outside of $C$. Since $q$ lies strictly outside of $C$, so must $d$ lie outside of $C$ as well, as $d$ lies on the extension of $s$ to $s'$. But this contradicts that $d$ intersects the line $e$ between $b$ and $c$, since $e$ is within $C$ but $d$ is outside of $C$. By contradiction, it must be that $\text{farthest}(p)$ is in $\text{CH}(X)$ for all $p$ in $P$.

**Note:** It is helpful to draw out for yourself the picture associated with this proof. Furthermore, note that the proof does not imply that both $b$ and $c$ must be farther than $q$ but that at least one of them must be so. Finally, it is easy to extend this proof to the pathological case that $q$ is on a hull boundary edge but not on the convex hull itself, as it will still remain true then by a similar argument that there is a farther point from $p$ than $q$. For that case, we would just have that $d = q$, and the ultimate contradiction still follows.

2. Assuming part (1) as true, prove that the farthest pair in $X$ belongs to $\text{CH}(X)$.

**Solution:** Let $p$ and $q$ be the farthest pair of points in $X$. This means that in particular that $p = \text{farthest}(q)$ and $q = \text{farthest}(p)$. By part (1), we have that $\text{farthest}(p)$ is in $\text{CH}(X)$ and $\text{farthest}(q)$ is in $\text{CH}(X)$, so that $p$ and $q$ must both be in $\text{CH}(X)$.
3. Now let us consider points on the convex hull only. List the (say $z$) points of $\text{CH}(X)$ starting from the leftmost point $p_1$ with the upper hull listed first clockwise as $\text{CH}(X) = \langle p_1, p_2, p_3, p_4, ... , p_z \rangle$.

Assume that there are at least 4 points in $\text{CH}(X)$ none of which are identical.

Prove that for any $j$, the following comparison holds true regarding the $j^{th}$ point $p_j$ and its consecutive clockwise neighbor $p_{(j+1) \text{ mod } z}$ along the convex hull:

If $p_s = \text{farthest}(p_j)$ and $p_t = \text{farthest}(p_{j+1})$, then $j \leq \text{ mod } z \ j+1 \leq \text{ mod } z \ s \leq \text{ mod } z \ t \leq \text{ mod } z \ j$.

Note that we can alternately phrase the implication as saying this: While going clockwise along $\text{CH}(X)$ starting from $p_j$, we hit $p_{j+1}$ before (or simultaneous with) hitting $p_s$ and then we hit $p_t$ before (or simultaneous with) returning to $p_j$.

**Solution:** Again, we make our argument via contradiction. Since $p_{j+1}$ by definition follows $p_j$ in the clockwise direction, the only meaningful negation of the statement we wish to prove is that $p_t$ lies strictly between $p_{j+1}$ and $p_s$ while going clockwise from $p_{j+1}$.

Meaning, for the sake of contradiction, we assume that the following ordering takes place (going clockwise): While going clockwise along $\text{CH}(X)$ starting from $p_j$, we hit $p_{j+1}$ before (or simultaneous with) hitting $p_t$ and then we hit $p_s$ before (or simultaneous with) returning to $p_j$. In particular, as all points lie on a convex point set, this means that a convex quadrilateral $Q$ is formed by the points $p_j$, $p_{j+1}$, $p_t$, and $p_s$ in that order. Recall that we have defined $p_s = \text{farthest}(p_j)$ and $p_t = \text{farthest}(p_{j+1})$. And, yet, the edges of our convex quadrilateral include the edges $e = (p_j, p_s)$ and $e' = (p_t, p_{j+1})$. Since one of these edges must be at least as long as the other, and it doesn’t matter which, take $e$ to be the longer of the two, so that $|e| \geq |e'|$. Now, consider the two diagonals of $Q$, $d = (p_j, p_t)$ and $d' = (p_{j+1}, p_s)$. By definition of farthest, it must be that the length of $e$ is greater than the length of $d$, and the length of $e'$ is greater than the length of $d'$, written as follows: $|e| > |d|$ and $|e'| > |d'|$. Combining with previous observation, we obtain also $|e| > |d'|$.

In other words, there is an edge, $e$, of our convex quadrilateral that is longer than both diagonals and its opposite edge $e'$. Again, returning to circles: Draw two circles $C$ and $C'$ such that both are of radius $|e|$, and $C$ is centered at point $p_j$ while $C'$ is centered at point $p_s$. The intersection of these two circles forms the lune $L(p_j, p_s)$. And, by observations above, our convex quadrilateral $Q$ must lie in the interior of $L(p_j, p_s)$. In fact, we may orient $Q$ and $L(p_j, p_s)$ so that $Q$ is exactly in the upper half of $L(p_j, p_s)$.
Denote the remaining edges of $Q$ as $f = (p_j, p_{j+1})$ and $g = (p_t, p_s)$. By definition of farthest, it must also be that $|e'| > |f|$ or else $p_j$ would be farthest($p_{j+1}$) instead of $p_t$. As we already noted that $|e'| > |d'|$, this means that the circle $C''$ centered at $p_{j+1}$ and with radius $|e'|$ must include both $p_j$ and $p_s$ strictly in its interior, with $p_t$ on its boundary.

We may orient $C''$ separately (not in the previous figure) to make $Q$ lie in the lower half. Now let angles $A$, $B$, $C$, and $D$ correspond to $Q$'s angles at $p_t$, $p_s$, $p_j$, and $p_{j+1}$ respectively. From the second figure it follows that $A$ is less than 90 degrees. It also follows that $B$ is greater than $C$ due to $|e| > |f|$, and $|e| > |e'|$. From the previous lune figure it follows that both $B$ and $C$ are less than 90 degrees and that at least one of them must be less than 60 degrees by convexity of $Q$. Therefore, $C$ is less than 60 degrees. Combining these observations and the fact that the sum of the angles of a quadrilateral add to 360 degrees we obtain that $D$ must be greater than 120 degrees, and therefore $D$ is the largest of the four angles. Looking at the triangle formed by edges $e'$, $d$, and $f$, ...
then this implies that the length of d is greatest in that triangle: \(|d| > |e'|\), and \(|d| > |f|\).

Combining (*): \(|e| > |d| > |e'| > \max(|d'|, |f|)\). Also, due to the lune, we have \(|e| > |g|\).

Further regarding angles, using the fact that we are in the lune, then the angle D' centered at \(p_{j+1}\) and formed by points \(p_j, p_{j+1},\) and \(p_s\) must be greater than 60 degrees. Similarly the angle A' centered at \(p_t\) and formed by points \(p_j, p_t,\) and \(p_s\) must be greater than 60 degrees. Here is a figure where the notation \(X = X' + X''\) is used for angle X.

Let's now re-state the angular inequalities so far, consistent with the figure above:

\[ C = C' + C'' < 60, \quad D = D' + D'' > 120, \quad A = A' + A'' < 90, \quad B = B' + B'' < 90, \quad D' > 60, \quad A' > 60. \]

Combining these, we further get that \(A'' < 30\), and, by the fact that \(|e| > \max(|d|, |d'|)\), we get that \(180 - T > 90\), so \(T < 90\). But \(T = C' + B' = D'' + A''\), and \(180 - T = C'' + D' = A' + B''\) so in particular \(D'' < 90\), and \(A' + B'' > 90\). Also, \(C' + C'' + B' < 120\) and also \(C' + B' + B'' < 120\) meaning that \(C' + B' < \min(120 - B'', 120 - C'')\). This means \(T < \min(120 - B'', 120 - C'')\).

Let \(m\) denote the side of triangle A', B'', T opposite from angle B'' (and of course \(m\) is part of diagonal d). Similarly, we let \(n\) denote the side of triangle D', C'', T opposite from angle C'' (and \(n\) is part of diagonal d'). Now, we will use the Law of sines: For triangle with sides a, b, c, opposite of angles A, B, and C, \(a/sinA = b/sinB = c/sinC\). So, combining the sine law with the inequalities (*) previously, we get the following:

\[ 1 < \frac{|e|}{|d|} = \frac{\sin(A')}{\sin(B)} \quad 1 < \frac{|e'|}{|d'|} = \frac{\sin(B'')}{\sin(A')} \]

This means that \(\sin(B'') > \sin(A)\) and also that \(\sin(A') > \sin(B)\) \(\Rightarrow\) Combining, \(\sin(A') + \sin(B'') > \sin(A) + \sin(B) = \sin(A' + A'') + \sin(B' + B'')\)
But, $60 < A' < A < 90$ and sine is strictly increasing in that interval, so $\sin(A) > \sin(A')$. This means that $(**) \sin(B'') > \sin(B) + \varepsilon$, where $\varepsilon > 0$. But, $0 < B'' < B < 90$ too, and, again, sine is monotonically increasing in the interval $[0, 90]$. Therefore, $\sin(B)$ must be greater than $\sin(B'')$. This contradicts inequality $(**)$. Therefore, by contradiction, we have established that the original claim of problem (3) must be true.

4. Assuming the observation in part (3) as true, construct an algorithm that, when given the convex hull $CH(X)$ as input, outputs both all pairs of the form $(p, \text{farthest}(p))$ for points $p$ in $CH(X)$ and runs in time linear in $h = |CH(X)|$ (the size of the convex hull). In particular, have your algorithm output the farthest pair in $X$ (without altering the asymptotic running time).

Solution:

FarthestPairs(point CH[1,2,…,r] : array holding the points of CH(X) in clockwise order) {
    int F[1,2,…,r]; // array that will have that CH[F[x]] is the farthest point from x
    int f = 1; double cur_dist, f_dist = 0;
    for (int i = 1 to r) do { // first find point farthest from CH[1]
        cur_dist = distance(CH[i],CH[1]); // sqrt((p1.x-p2.x)+sqr(p1.y-p2.y))
        if (cur_dist > f_dist) { f = i; f_dist = cur_dist; } // update index and distance
    } // at the end of this loop, we just discover the point farthest from p_1
    F[1] = f; // f is now the index of the farthest point from the first point
    int max1 = 1, max2 = f; // initialize farthest pair too
    for (int i = 2 to r) do { // we now compute the remainder of array F
        cur_dist = distance(CH[i],CH[f]); // initialize
        while (cur_dist < distance(CH[i], CH[f+1]) do {
            f++;// representing a cw rotation of the farthest point candidate
        } // we rotate the candidate until CH[f] is farther to p_i than CH[f+1]
        F[i] = f; // f is the farthest point to p_i now
        if (cur_dist > f_dist) { f_dist = cur_dist; max1 = i; max2 = f; }
    } // now we computed array F as well as the farthest pair max1, max2
    Output(F, max1, max2); }