Chapter 2

Mathematics of Cryptography

Part I: Modular Arithmetic, Congruence, and Matrices

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Objectives

- ❑ To review integer arithmetic, concentrating on divisibility and finding the greatest common divisor using the Euclidean algorithm
- ❑ To understand how the extended Euclidean algorithm can be used to solve linear Diophantine equations, to solve linear congruent equations, and to find the multiplicative inverses
- ❑ To emphasize the importance of modular arithmetic and the modulo operator, because they are extensively used in cryptography
- ❑ To emphasize and review matrices and operations on residue matrices that are extensively used in cryptography
- ❑ To solve a set of congruent equations using residue matrices

2-1 INTEGER ARITHMETIC

In integer arithmetic, we use a set and a few operations. You are familiar with this set and the corresponding operations, but they are reviewed here to create a background for modular arithmetic.

Topics discussed in this section:

2.1.1 Set of Integers
2.1.2 Binary Operations
2.1.3 Integer Division
2.1.4 Divisibility
2.1.5 Linear Diophantine Equations

2.1.1 Set of Integers

The set of integers, denoted by Z, contains all integral numbers (with no fraction) from negative infinity to positive infinity (Figure 2.1).

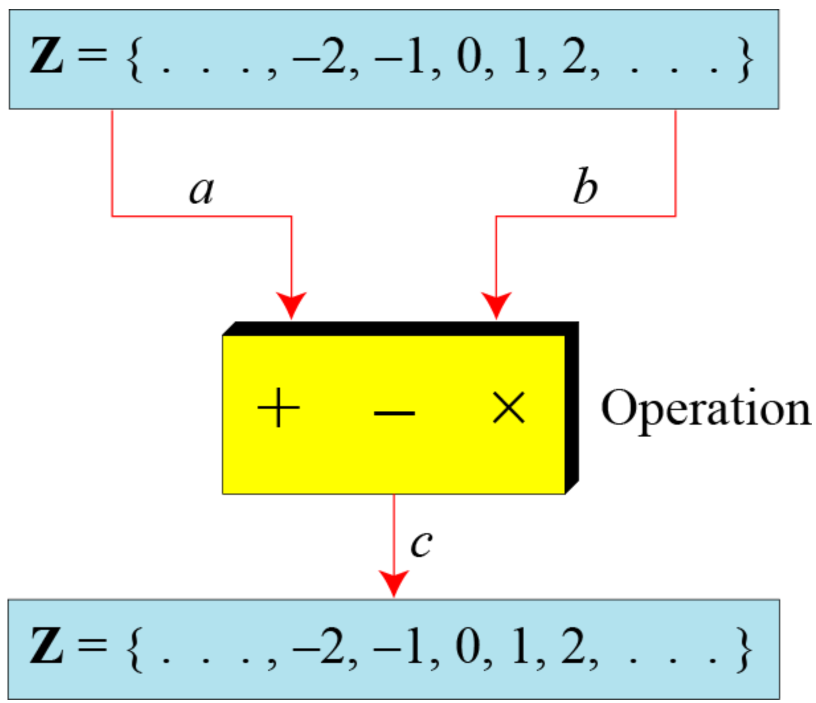
Figure 2.1 ***The set of integers***

$$\mathbf{Z} = \{\ \ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\ \}$$

2.1.2 Binary Operations

In cryptography, we are interested in three binary operations applied to the set of integers. A binary operation takes two inputs and creates one output.

Figure 2.2 ***Three binary operations for the set of integers***



2.1.2 *Continued*

Example 2.1

The following shows the results of the three binary operations on two integers. Because each input can be either positive or negative, we can have four cases for each operation.

Add:	$5 + 9 = 14$	$(-5) + 9 = 4$	$5 + (-9) = -4$	$(-5) + (-9) = -14$
Subtract:	$5 - 9 = -4$	$(-5) - 9 = -14$	$5 - (-9) = 14$	$(-5) - (-9) = +4$
Multiply:	$5 \times 9 = 45$	$(-5) \times 9 = -45$	$5 \times (-9) = -45$	$(-5) \times (-9) = 45$

2.1.3 Integer Division

In integer arithmetic, if we divide a by n, we can get q And r . The relationship between these four integers can be shown as

$$a = q \times n + r$$

2.1.3 Continued

Example 2.2

Assume that a = 255 and n = 11. We can find q = 23 and R = 2 using the division algorithm.

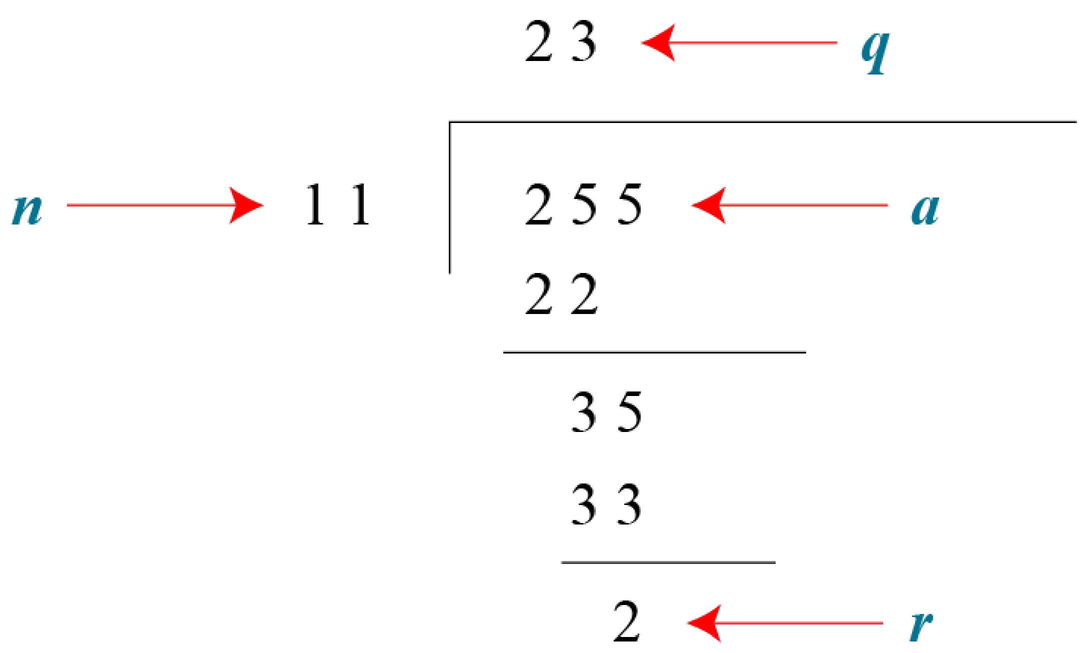
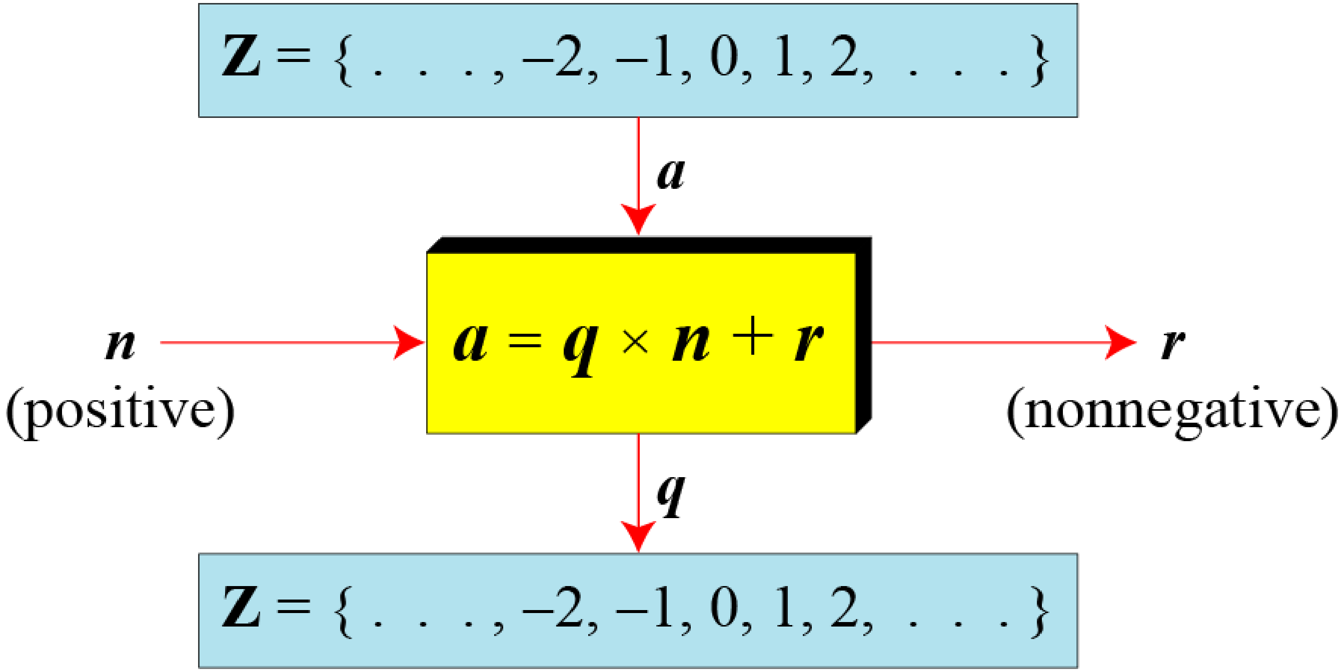


Figure 2.3 ***Example 2.2, finding the quotient and the remainder***

2.1.3 Continued

Figure 2.4 *Division algorithm for integers*



2.1.3 Continued

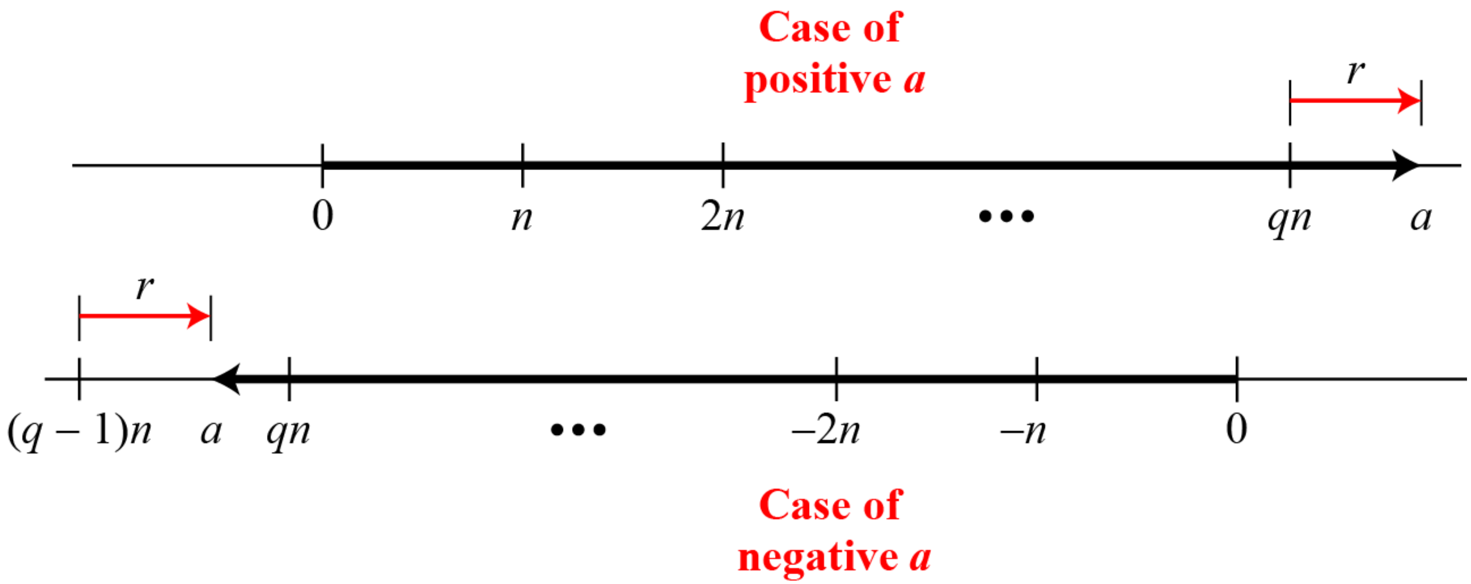
Example 2.3

When we use a computer or a calculator, *r* and *q* are negative when *a* is negative. How can we apply the restriction that *r* needs to be positive? The solution is simple, we decrement the value of *q* by 1 and we add the value of *n* to *r* to make it positive.

$$-255 = (-\mathbf{23} \times 11) + (-\mathbf{2}) \quad \leftrightarrow \quad -255 = (-\mathbf{24} \times 11) + \mathbf{9}$$

2.1.3 Continued

Figure 2.5 ***Graph of division algorithm***



2.1.4 Divisibility

If a is not zero and we let $r = 0$ in the division relation, we get

$$a = q \times n$$

If the remainder is zero, $a \mid n$

If the remainder is not zero, $a \nmid n$

2.1.4 Continued

Example 2.4

a. The integer 4 divides the integer 32 because 32 = 8 × 4. We show this as

$$4 \mid 32$$

b. The number 8 does not divide the number 42 because 42 = 5 × 8 + 2. There is a remainder, the number 2, in the equation. We show this as

$$8 \nmid 42$$

2.1.4 Continued

Properties

Property 1: if $a|1$, then $a = \pm 1$.

Property 2: if $a|b$ and $b|a$, then $a = \pm b$.

Property 3: if $a|b$ and $b|c$, then $a|c$.

Property 4: if $a|b$ and $a|c$, then $a|(m \times b + n \times c)$, where m and n are arbitrary integers

2.1.4 *Continued*

Example 2.5

a. We have $13 \mid 78$, $7 \mid 98$, $-6 \mid 24$, $4 \mid 44$, and $11 \mid (-33)$.

b. We have $13 \nmid 27$, $7 \nmid 50$, $-6 \nmid 23$, $4 \nmid 41$, and $11 \nmid (-32)$.

2.1.4 *Continued*

Example 2.6

a. Since $3|15$ and $15|45$,

according to the third property, $3|45$.

b. Since $3|15$ and $3|9$,

according to the fourth property,

$3|(15 \times 2 + 9 \times 4)$, which means $3|66$.

2.1.4 Continued

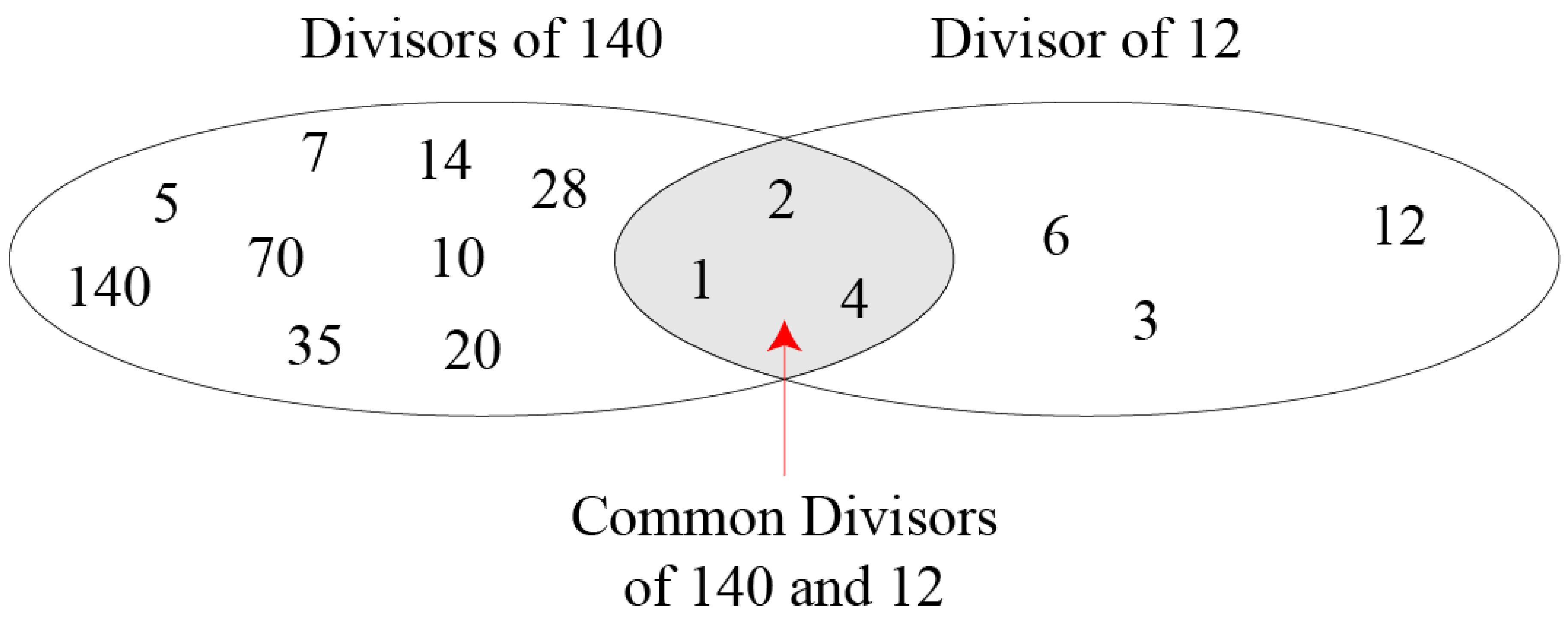
Note

Fact 1: The integer 1 has only one divisor, itself.

Fact 2: Any positive integer has at least two divisors, 1 and itself (but it can have more).

2.1.4 Continued

Figure 2.6 ***Common divisors of two integers***



2.1.4 Continued

Note Greatest Common Divisor

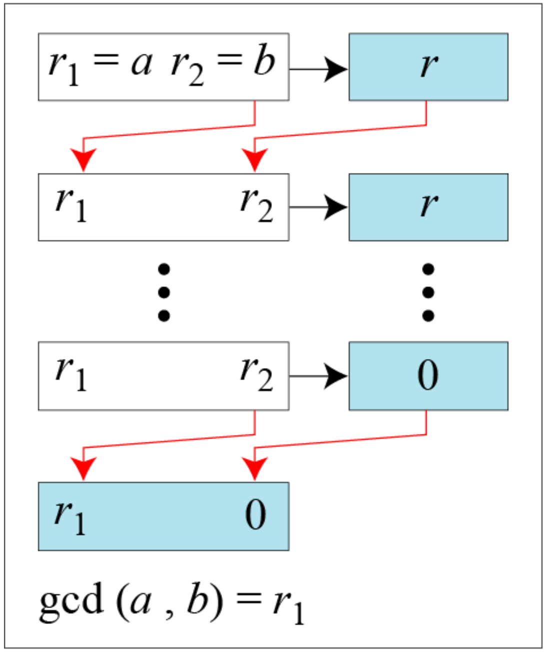
The greatest common divisor of two positive integers is the largest integer that can divide both integers.

Note Euclidean Algorithm

Fact 1: gcd (a, 0) = a
Fact 2: gcd (a, b) = gcd (b, r), where r is the remainder of dividing a by b

2.1.4 Continued

Figure 2.7 ***Euclidean Algorithm***



a. Process

$r_1 \leftarrow a; \quad r_2 \leftarrow b;$ (Initialization)

while ($r_2 > 0$)

{

$q \leftarrow r_1 / r_2;$

$r \leftarrow r_1 - q \times r_2;$

$r_1 \leftarrow r_2; \quad r_2 \leftarrow r;$

}

gcd (a, b) $\leftarrow r_1$

b. Algorithm

Note

When gcd (a, b) = 1, we say that a and b are relatively prime.

2.1.4 Continued

Note

When gcd (a, b) = 1, we say that a and b are relatively prime.

2.1.4 Continued

Example 2.7

Find the greatest common divisor of 2740 and 1760.

Solution

We have gcd (2740, 1760) = 20.

q	r_1	r_2	r
1	2740	1760	980
1	1760	980	780
1	980	780	200
3	780	200	180
1	200	180	20
9	180	20	0
	20	0	

2.1.4 *Continued*

Example 2.8

Find the greatest common divisor of 25 and 60.

Solution

We have gcd (25, 65) = 5.

q	r_1	r_2	r
0	25	60	25
2	60	25	10
2	25	10	5
2	10	5	0
	5	0	

2.1.4 Continued

Extended Euclidean Algorithm

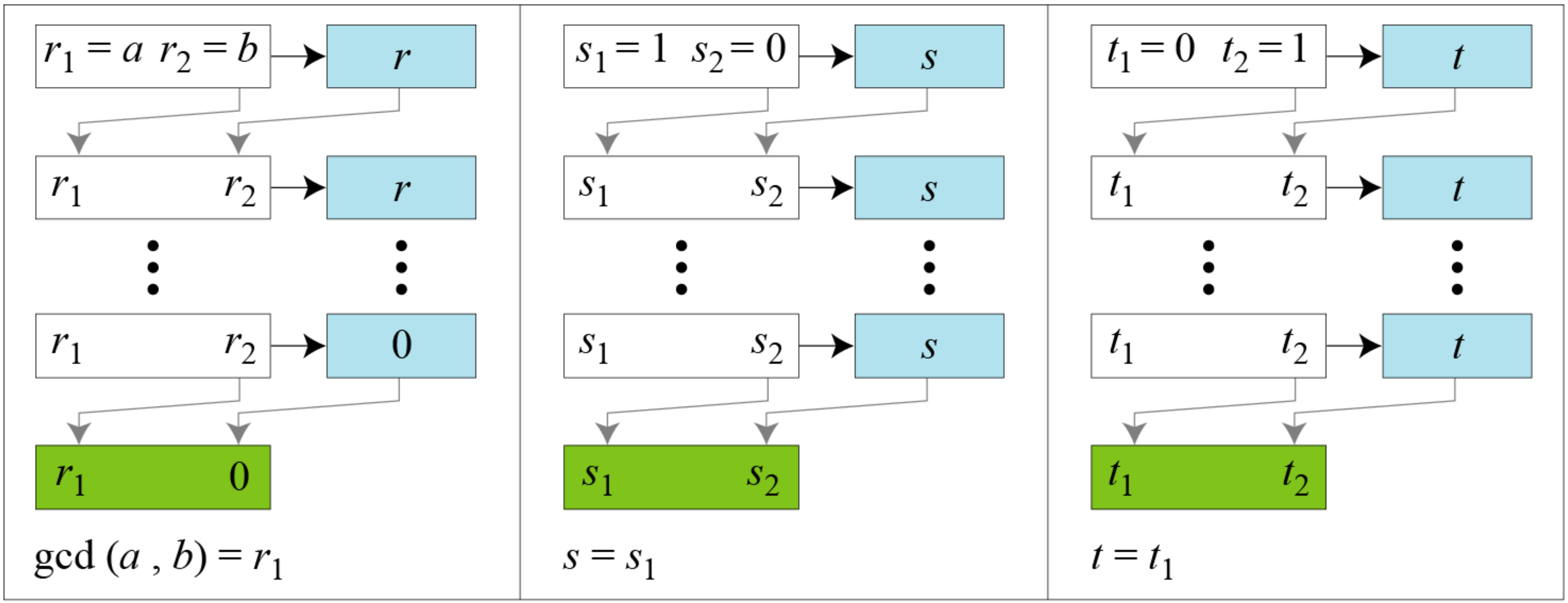
Given two integers *a* and *b*, we often need to find other two integers, *s* and *t*, such that

$$s \times a + t \times b = \gcd(a, b)$$

The extended Euclidean algorithm can calculate the gcd (*a*, *b*) and at the same time calculate the value of *s* and *t*.

2.1.4 Continued

Figure 2.8.a ***Extended Euclidean algorithm, part a***



a. Process

2.1.4 Continued

Figure 2.8.b *Extended Euclidean algorithm, part b*

$r_1 \leftarrow a;$ $r_2 \leftarrow b;$
$s_1 \leftarrow 1;$ $s_2 \leftarrow 0;$ (Initialization)
$t_1 \leftarrow 0;$ $t_2 \leftarrow 1;$

while ($r_2 > 0$)
{
$q \leftarrow r_1 / r_2;$

$r \leftarrow r_1 - q \times r_2;$
$r_1 \leftarrow r_2;$ $r_2 \leftarrow r;$ (Updating r's)

$s \leftarrow s_1 - q \times s_2;$
$s_1 \leftarrow s_2;$ $s_2 \leftarrow s;$ (Updating s's)

$t \leftarrow t_1 - q \times t_2;$
$t_1 \leftarrow t_2;$ $t_2 \leftarrow$ t; (Updating t's)
}

gcd (a , b) $\leftarrow r_1;$ $s \leftarrow s_1;$ $t \leftarrow t_1$

b. Algorithm

2.1.4 Continued

Example 2.9

Given *a* = 161 and *b* = 28, find gcd (*a*, *b*) and the values of *s* and *t*.

Solution

We get gcd (161, 28) = 7, *s* = −1 and *t* = 6.

q	r_1	r_2	r	s_1	s_2	s	t_1	t_2	t
5	161	28	21	1	0	1	0	1	−5
1	28	21	7	0	1	−1	1	−5	6
3	21	7	0	1	−1	4	−5	6	−23
	7	0		**−1**	4		**6**	−23	

2.1.4 Continued

Example 2.10

Given *a* = 17 and *b* = 0, find gcd (*a*, *b*) and the values of *s* and *t*.

Solution

We get gcd (17, 0) = 17, *s* = 1, and *t* = 0.

q	r_1	r_2	r	s_1	s_2	s	t_1	t_2	t
	17	0		1	0		0	1	

2.1.4 Continued

Example 2.11

Given a = 0 and b = 45, find gcd (a, b) and the values of s and t.

Solution

We get gcd (0, 45) = 45, s = 0, and t = 1.

q	r_1	r_2	r	s_1	s_2	s	t_1	t_2	t
0	0	45	0	1	0	1	0	1	0
	45	0		0	1		**1**	0	

2.1.4 Continued

Linear Diophantine Equation

Note

A linear Diophantine equation of two variables is $ax + by = c$.

Note

Particular solution:

$x_0 = (c/d)s$ and $y_0 = (c/d)t$

Note

General solutions:

$x = x_0 + k\,(b/d)$ and $y = y_0 - k(a/d)$

where k is an integer

2.1.4 *Continued*

Example 2.12

Find the particular and general solutions to the equation $21x + 14y = 35$.

Solution

$$\text{Particular: } x_0 = 5 \times 1 = 5 \quad \text{and} \quad y_0 = 5 \times (-1) = -5$$
$$\text{General: } x = 5 + k \times 2 \quad \text{and} \quad y = -5 - k \times 3$$

2.1.4 Continued

Example 2.13

For example, imagine we want to cash a $100 check and get some $20 and some $5 bills. We have many choices, which we can find by solving the corresponding Diophantine equation $20x + 5y = 100$. Since $d = \gcd(20, 5) = 5$ and $5 \mid 100$, the equation has an infinite number of solutions, but only a few of them are acceptable in this case The general solutions with x and y nonnegative are

(0, 20), (1, 16), (2, 12), (3, 8), (4, 4), (5, 0).

2-2 MODULAR ARITHMETIC

The division relationship ($a = q \times n + r$) discussed in the previous section has two inputs (a and n) and two outputs (q and r). In modular arithmetic, we are interested in only one of the outputs, the remainder r.

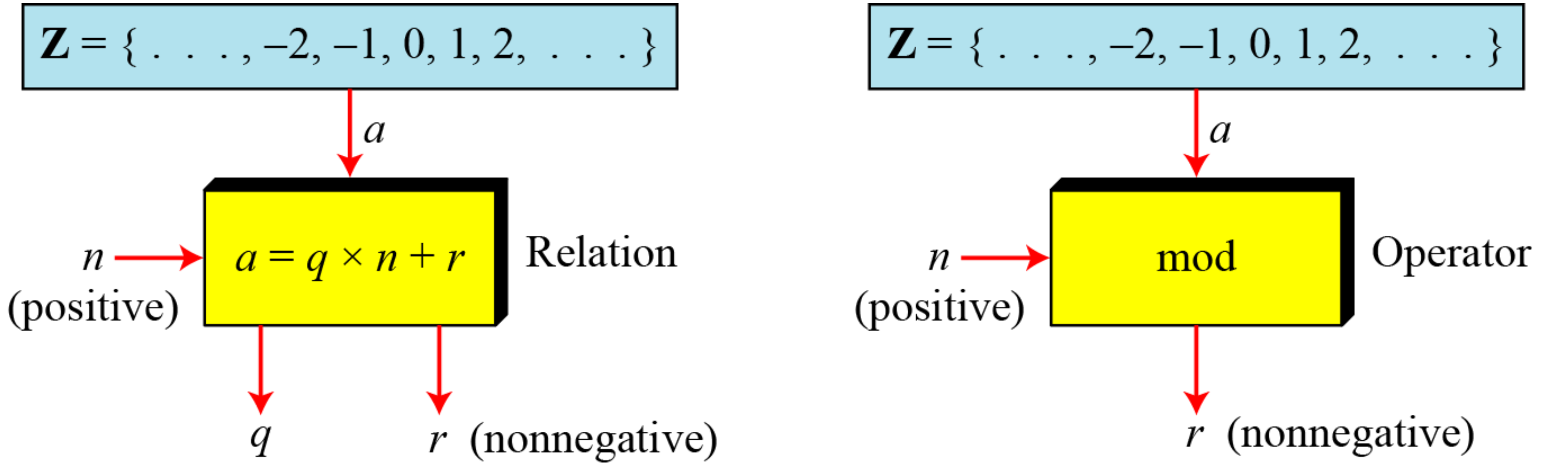
Topics discussed in this section:

2.2.1 Modular Operator
2.2.2 Set of Residues
2.2.3 Congruence
2.2.4 Operations in Z_n
2.2.5 Addition and Multiplication Tables
2.2.6 Different Sets

2.2.1 Modulo Operator

The modulo operator is shown as mod. The second input (n) is called the modulus. The output r is called the residue.

Figure 2.9 ***Division algorithm and modulo operator***



Example 2.14

Find the result of the following operations:

a. **27 mod 5**

b. **36 mod 12**

c. **−18 mod 14**

d. **−7 mod 10**

Solution

a. **Dividing 27 by 5 results in *r* = 2**

b. **Dividing 36 by 12 results in *r* = 0.**

c. **Dividing −18 by 14 results in *r* = −4. After adding the modulus *r* = 10**

d. **Dividing −7 by 10 results in r = −7. After adding the modulus to −7, r = 3.**

2.2.2 Set of Residues

The modulo operation creates a set, which in modular arithmetic is referred to as the set of least residues modulo n, or Z_n.

Figure 2.10 ***Some Z_n sets***

$$\mathbf{Z}_n = \{ 0, 1, 2, 3, \ldots, (n-1) \}$$

$$\mathbf{Z}_2 = \{ 0, 1 \} \quad \mathbf{Z}_6 = \{ 0, 1, 2, 3, 4, 5 \} \quad \mathbf{Z}_{11} = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$$

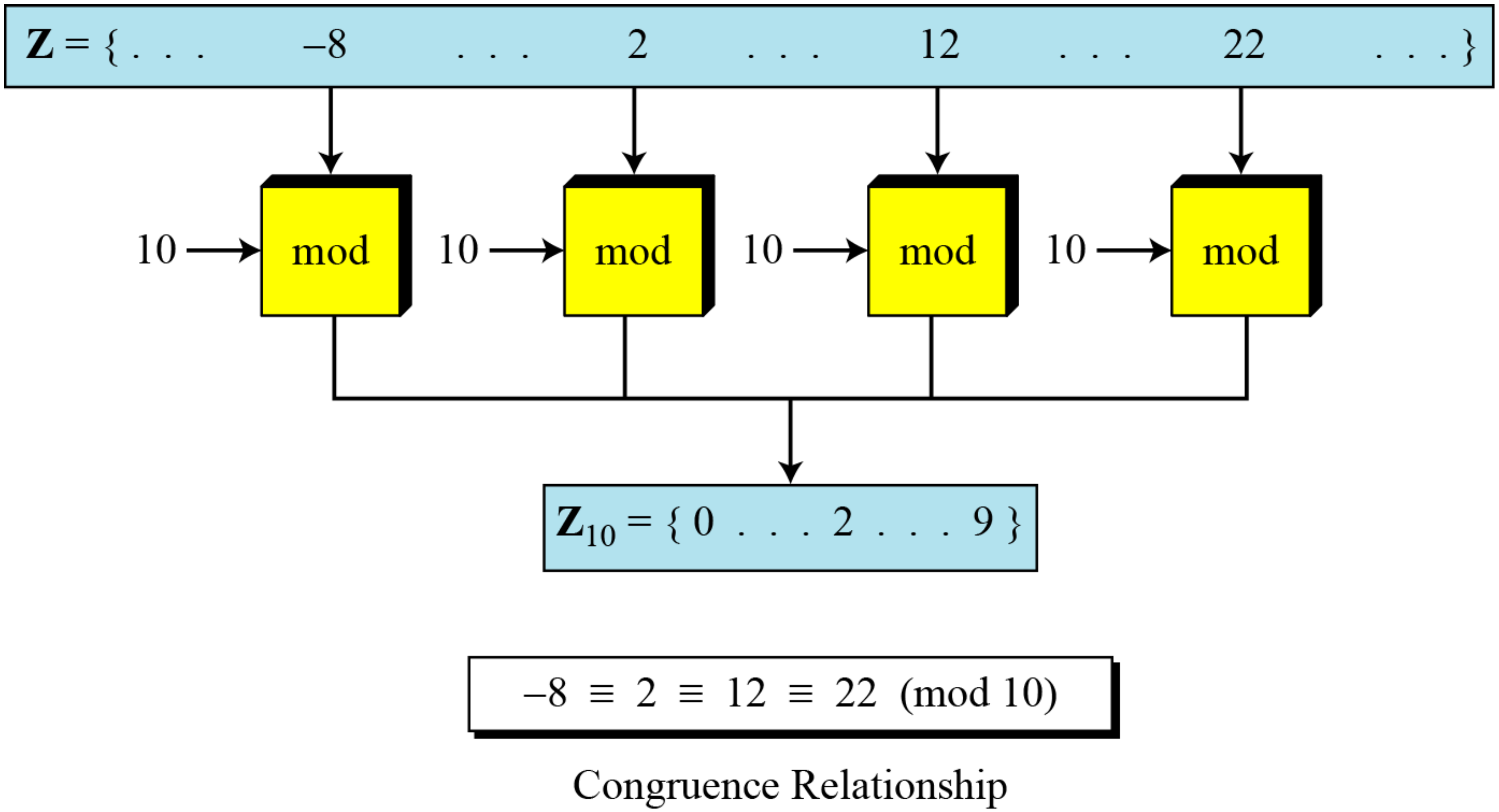
2.2.3 Congruence

To show that two integers are congruent, we use the congruence operator (≡). For example, we write:

$2 \equiv 12 \pmod{10}$	$13 \equiv 23 \pmod{10}$
$3 \equiv 8 \pmod{5}$	$8 \equiv 13 \pmod{5}$

2.2.3 Continued

Figure 2.11 ***Concept of congruence***



2.2.3 Continued

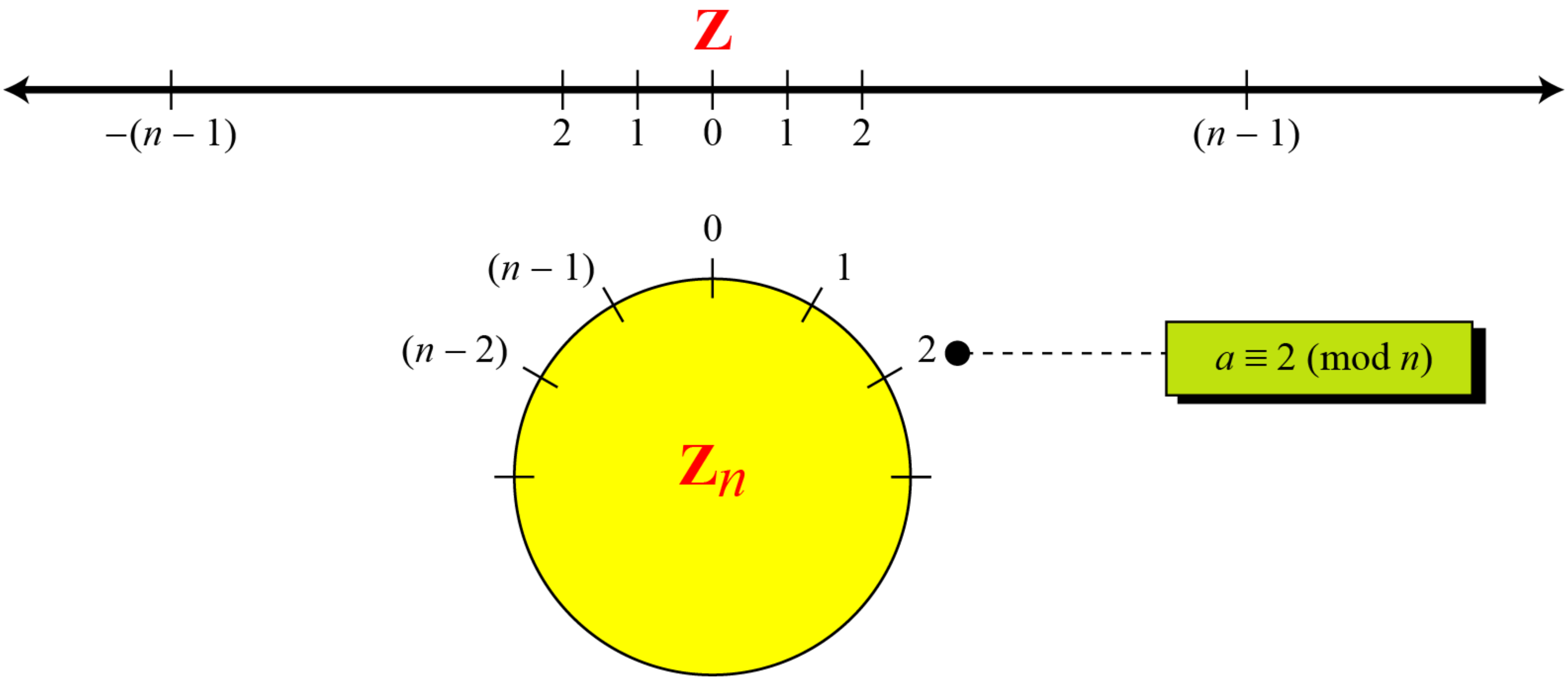
Residue Classes

A residue class [a] or $[a]_n$ is the set of integers congruent modulo n.

$$[0] = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$$
$$[1] = \{\ldots, -14, -9, -4, 1, 6, 11, 16, \ldots\}$$
$$[2] = \{\ldots, -13, -8, -3, 2, 7, 12, 17, \ldots\}$$
$$[3] = \{\ldots, -12, -7, -5, 3, 8, 13, 18, \ldots\}$$
$$[4] = \{\ldots, -11, -6, -1, 4, 9, 14, 19, \ldots\}$$

2.2.3 Continued

Figure 2.12 ***Comparison of Z and Z_n using graphs***



2.2.3 Continued

Example 2.15

We use modular arithmetic in our daily life; for example, we use a clock to measure time. Our clock system uses modulo 12 arithmetic. However, instead of a 0 we use the number 12.

2.2.4 Operation in Z_n

The three binary operations that we discussed for the set Z can also be defined for the set Zn. The result may need to be mapped to Zn using the mod operator.

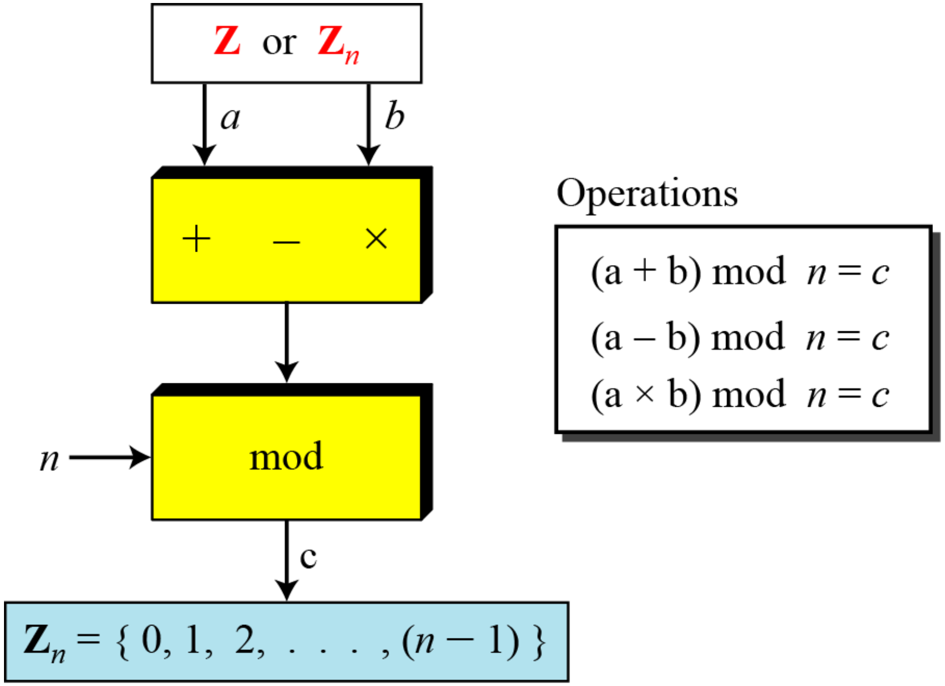


Figure 2.13 *Binary operations in Z_n*

2.2.4 *Continued*

Example 2.16

Perform the following operations (the inputs come from Zn):
a. Add 7 to 14 in Z15.
b. Subtract 11 from 7 in Z13.
c. Multiply 11 by 7 in Z20.

Solution

$$(14 + 7) \bmod 15 \rightarrow (21) \bmod 15 = 6$$
$$(7 - 11) \bmod 13 \rightarrow (-4) \bmod 13 = 9$$
$$(7 \times 11) \bmod 20 \rightarrow (77) \bmod 20 = 17$$

Example 2.17

Perform the following operations (the inputs come from either Z or Z_n):

a. Add 17 to 27 in Z_{14}.

b. Subtract 43 from 12 in Z_{13}.

c. Multiply 123 by −10 in Z_{19}.

Solution

$$(14 + 7) \bmod 15 \rightarrow (21) \bmod 15 = 6$$

$$(7 - 11) \bmod 13 \rightarrow (-4) \bmod 13 = 9$$

$$(7 \times 11) \bmod 20 \rightarrow (77) \bmod 20 = 17$$

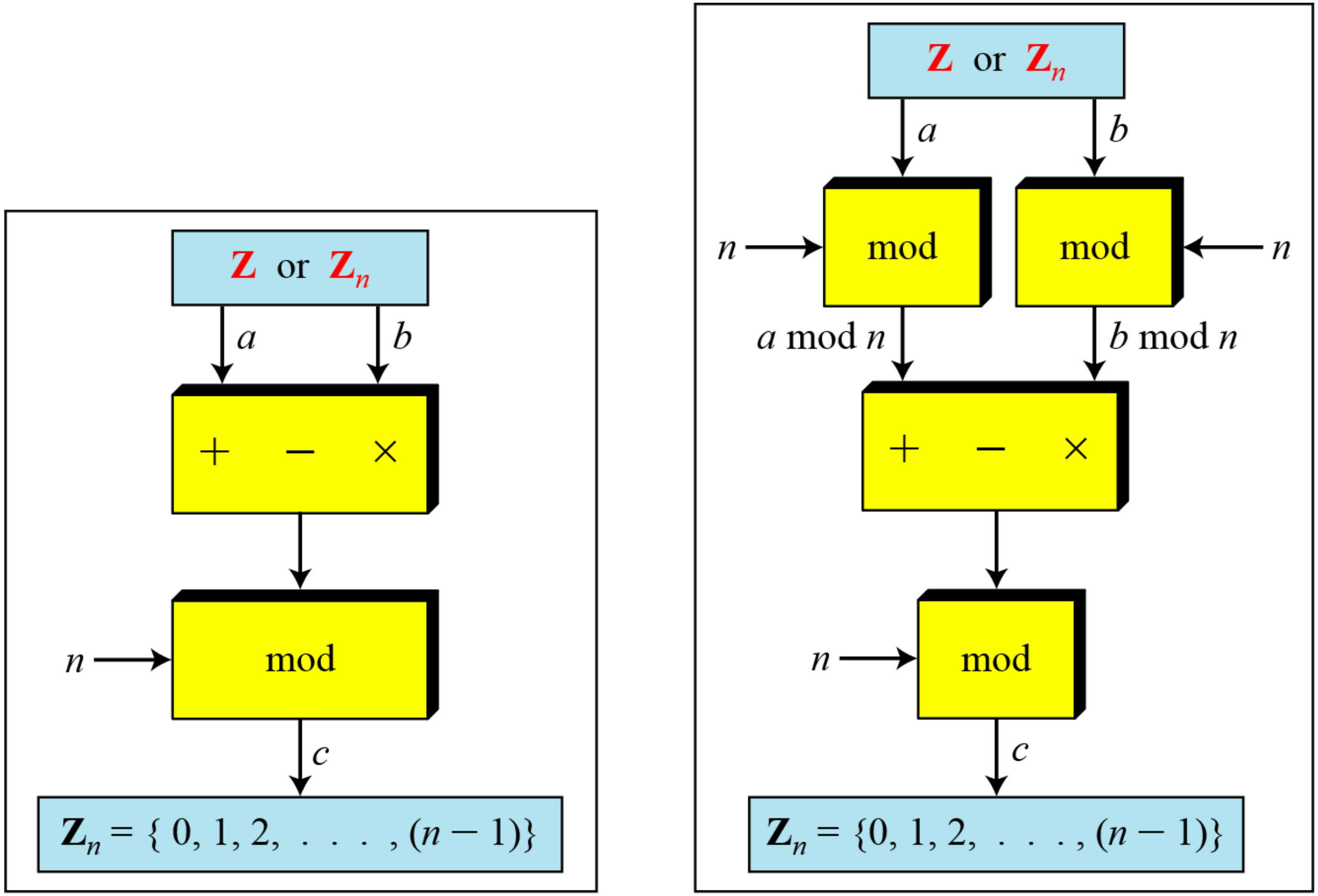
2.2.4 Continued

Properties

First Property:	$(a + b) \bmod n = [(a \bmod n) + (b \bmod n)] \bmod n$
Second Property:	$(a - b) \bmod n = [(a \bmod n) - (b \bmod n)] \bmod n$
Third Property:	$(a \times b) \bmod n = [(a \bmod n) \times (b \bmod n)] \bmod n$

2.2.4 Continued

Figure 2.14 ***Properties of mode operator***



a. Original process

b. Applying properties

2.2.4 Continued

Example 2.18

The following shows the application of the above properties:

1. (1,723,345 + 2,124,945) mod 11 = (8 + 9) mod 11 = 6

2. (1,723,345 − 2,124,945) mod 16 = (8 − 9) mod 11 = 10

3. (1,723,345 × 2,124,945) mod 16 = (8 × 9) mod 11 = 6

2.2.4 Continued

Example 2.19

In arithmetic, we often need to find the remainder of powers of 10 when divided by an integer.

$$10^n \bmod x = (10 \bmod x)^n \qquad \text{Applying the third property } n \text{ times.}$$

$$10 \bmod 3 = 1 \quad \rightarrow \quad 10^n \bmod 3 = (10 \bmod 3)^n = 1$$

$$10 \bmod 9 = 1 \quad \rightarrow \quad 10^n \bmod 9 = (10 \bmod 9)^n = 1$$

$$10 \bmod 7 = 3 \quad \rightarrow \quad 10^n \bmod 7 = (10 \bmod 7)^n = 3^n \bmod 7$$

Example 2.20

We have been told in arithmetic that the remainder of an integer divided by 3 is the same as the remainder of the sum of its decimal digits. We write an integer as the sum of its digits multiplied by the powers of 10.

$$a = a_n \times 10^n + \cdots + a_1 \times 10^1 + a_0 \times 10^0$$

For example: $6371 = 6 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 1 \times 10^0$

$$\begin{aligned} a \bmod 3 &= (a_n \times 10^n + \cdots + a_1 \times 10^1 + a_0 \times 10^0) \bmod 3 \\ &= (a_n \times 10^n) \bmod 3 + \cdots + (a_1 \times 10^1) \bmod 3 + (a_0 \times 10^0) \bmod 3 \\ &= (a_n \bmod 3) \times (10^n \bmod 3) + \cdots + (a_1 \bmod 3) \times (10^1 \bmod 3) + \\ &\quad (a_0 \bmod 3) \times (10^0 \bmod 3) \\ &= a_n \bmod 3 + \cdots + a_1 \bmod 3 + a_0 \bmod 3 \\ &= (a_n + \cdots + a_1 + a_0) \bmod 3 \end{aligned}$$

2.2.5 *Inverses*

When we are working in modular arithmetic, we often need to find the inverse of a number relative to an operation. We are normally looking for an additive inverse (relative to an addition operation) or a multiplicative inverse (relative to a multiplication operation).

2.2.5 Continue

Additive Inverse

In Z_n, two numbers *a* and *b* are additive inverses of each other if

$$a + b \equiv 0 \pmod{n}$$

Note

In modular arithmetic, each integer has an additive inverse. The sum of an integer and its additive inverse is congruent to 0 modulo n.

2.2.5 *Continued*

Example 2.21

Find all additive inverse pairs in Z10.

Solution

The six pairs of additive inverses are (0, 0), (1, 9), (2, 8), (3, 7), (4, 6), and (5, 5).

2.2.5 Continue

Multiplicative Inverse

In Z_n, two numbers *a* and *b* are the multiplicative inverse of each other if

$$a \times b \equiv 1 \pmod{n}$$

Note

In modular arithmetic, an integer may or may not have a multiplicative inverse. When it does, the product of the integer and its multiplicative inverse is congruent to 1 modulo n.

Example 2.22

Find the multiplicative inverse of 8 in Z_{10}.

Solution

There is no multiplicative inverse because gcd (10, 8) = 2 ≠ 1. In other words, we cannot find any number between 0 and 9 such that when multiplied by 8, the result is congruent to 1.

Example 2.23

Find all multiplicative inverses in Z_{10}.

Solution

There are only three pairs: (1, 1), (3, 7) and (9, 9). The numbers 0, 2, 4, 5, 6, and 8 do not have a multiplicative inverse.

Example 2.24

Find all multiplicative inverse pairs in Z_{11}.

Solution

We have seven pairs: (1, 1), (2, 6), (3, 4), (5, 9), (7, 8), (9, 9), and (10, 10).

2.2.5 Continued

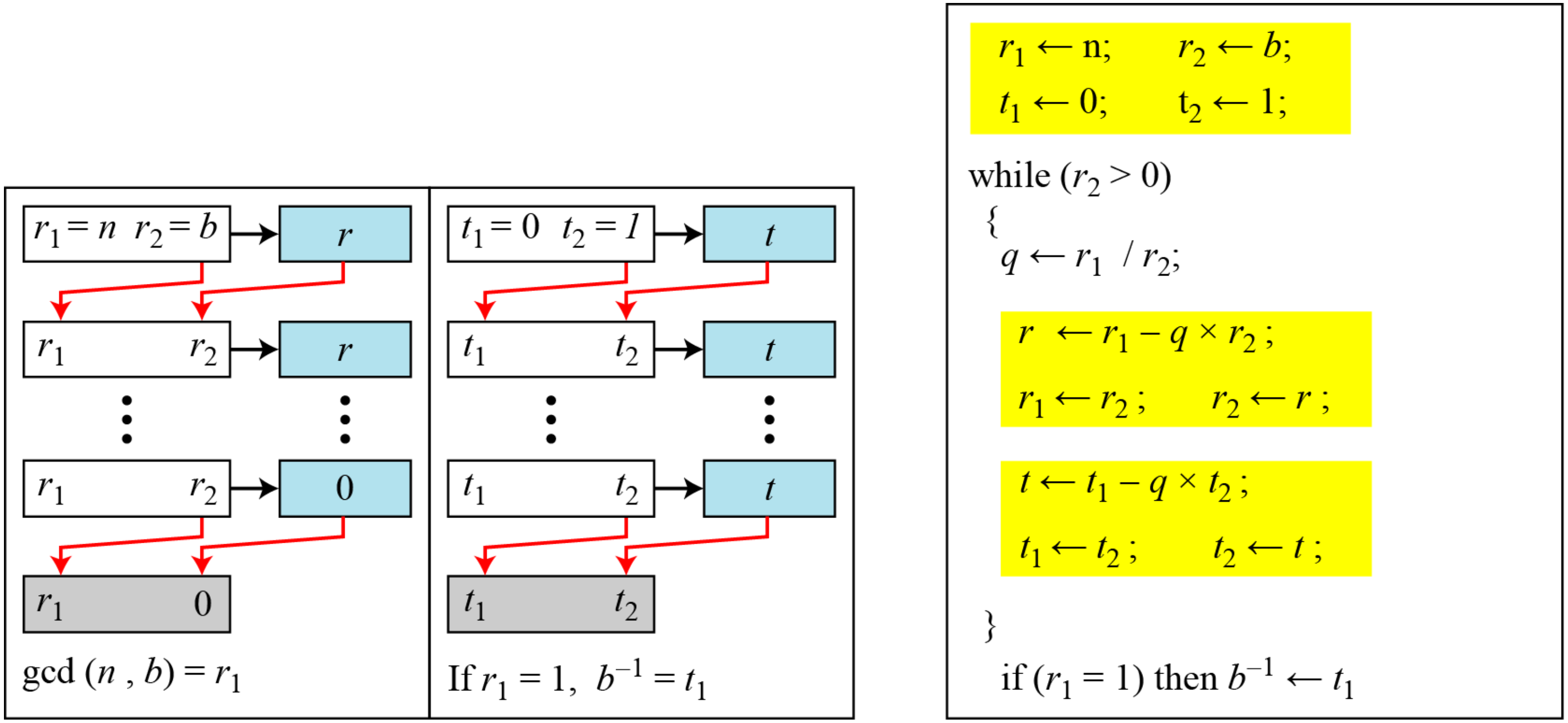
Note

The extended Euclidean algorithm finds the multiplicative inverses of b in Z_n when n and b are given and gcd (n, b) = 1.

The multiplicative inverse of b is the value of t after being mapped to Z_n.

2.2.5 Continued

Figure 2.15 *Using extended Euclidean algorithm to find multiplicative inverse*



a. Process b. Algorithm

2.2.5 *Continued*

Example 2.25

Find the multiplicative inverse of 11 in Z_{26}.

Solution

q	r_1	r_2	r	t_1	t_2	t
2	26	11	4	0	1	–2
2	11	4	3	1	–2	5
1	4	3	1	–2	5	–7
3	3	1	0	5	–7	26
	1	0		–7	26	

The gcd (26, 11) is 1; the inverse of 11 is –7 or 19.

2.2.5 Continued

Example 2.26

Find the multiplicative inverse of 23 in Z_{100}.

Solution

q	r_1	r_2	r	t_1	t_2	t
4	100	23	8	0	1	–4
2	23	8	7	1	–4	19
1	8	7	1	–4	9	–13
7	7	1	0	9	–13	100
	1	0		–13	100	

The gcd (100, 23) is 1; the inverse of 23 is –13 or 87.

2.2.5 Continued

Example 2.27

Find the inverse of 12 in Z_{26}.

Solution

q	r_1	r_2	r	t_1	t_2	t
2	26	12	2	0	1	−2
6	12	2	0	1	−2	13
	2	0		−2	13	

The gcd (26, 12) is 2; the inverse does not exist.

Figure 2.16 ***Addition and multiplication table for Z_{10}***

	0	1	2	3	4	5	6	7	8	9
0	**0**	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	**0**
2	2	3	4	5	6	7	8	9	**0**	1
3	3	4	5	6	7	8	9	**0**	1	2
4	4	5	6	7	8	9	**0**	1	2	3
5	5	6	7	8	9	**0**	1	2	3	4
6	6	7	8	9	**0**	1	2	3	4	5
7	7	8	9	**0**	1	2	3	4	5	6
8	8	9	**0**	1	2	3	4	5	6	7
9	9	**0**	1	2	3	4	5	6	7	8

Addition Table in $\mathbf{Z}_{10}$

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	**1**	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	**1**	8	0	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	**1**

Multiplication Table in $\mathbf{Z}_{10}$

2.2.7 Different Sets

Figure 2.17 ***Some Z_n and Z_n* sets***

$Z_6 = \{0, 1, 2, 3, 4, 5\}$

$Z_6^* = \{1, 5\}$

$Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$Z_7^* = \{1, 2, 3, 4, 5, 6\}$

$Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$Z_{10}^* = \{1, 3, 7, 9\}$

Note

We need to use Zn when additive inverses are needed; we need to use Zn* when multiplicative inverses are needed.

2.2.8 Two More Sets

Cryptography often uses two more sets: Z_p and Z_p*. The modulus in these two sets is a prime number.

$$Z_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$
$$Z_{13}{}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

2-3 MATRICES

In cryptography we need to handle matrices. Although this topic belongs to a special branch of algebra called linear algebra, the following brief review of matrices is necessary preparation for the study of cryptography.

Topics discussed in this section:

2.3.1 Definitions
2.3.2 Operations and Relations
2.3.3 Determinants
2.3.4 Residue Matrices

2.3.1 Definition

Figure 2.18 ***A matrix of size l × m***

m columns

$$\text{Matrix } \mathbf{A}: \quad l \text{ rows} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} \end{bmatrix}$$

2.3.1 *Continued*

Figure 2.19 ***Examples of matrices***

$$\begin{bmatrix} 2 & 1 & 5 & 11 \end{bmatrix}$$

Row matrix

$$\begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

Column matrix

$$\begin{bmatrix} 23 & 14 & 56 \\ 12 & 21 & 18 \\ 10 & 8 & 31 \end{bmatrix}$$

Square matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

0

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I

Example 2.28

Figure 2.20 shows an example of addition and subtraction.

Figure 2.20 ***Addition and subtraction of matrices***

$$\begin{bmatrix} 12 & 4 & 4 \\ 11 & 12 & 30 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 3 & 2 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 2 & 3 \\ 8 & 10 & 20 \end{bmatrix}$$

C = A + B

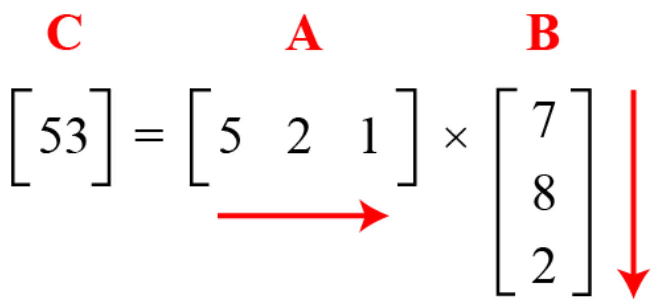
$$\begin{bmatrix} -2 & 0 & -2 \\ -5 & -8 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 3 & 2 & 10 \end{bmatrix} - \begin{bmatrix} 7 & 2 & 3 \\ 8 & 10 & 20 \end{bmatrix}$$

D = A − B

Example 2. 29

Figure 2.21 shows the product of a row matrix (1 × 3) by a column matrix (3 × 1). The result is a matrix of size 1 × 1.

Figure 2.21 *Multiplication of a row matrix by a column matrix*



In which: $53 = 5 \times 7 + 2 \times 8 + 1 \times 2$

Example 2. 30

Figure 2.22 shows the product of a 2 × 3 matrix by a 3 × 4 matrix. The result is a 2 × 4 matrix.

Figure 2.22 *Multiplication of a 2 × 3 matrix by a 3 × 4 matrix*

$$\underset{\mathbf{C}}{\begin{bmatrix} 52 & 18 & 14 & 9 \\ 41 & 21 & 22 & 7 \end{bmatrix}} = \underset{\mathbf{A}}{\begin{bmatrix} 5 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}} \times \underset{\mathbf{B}}{\begin{bmatrix} 7 & 3 & 2 & 1 \\ 8 & 0 & 0 & 2 \\ 1 & 3 & 4 & 0 \end{bmatrix}}$$

2.3.2 *Continued*

Example 2. 31

Figure 2.23 shows an example of scalar multiplication.

Figure 2.23 ***Scalar multiplication***

$$\underset{\mathbf{B}}{\begin{bmatrix} 15 & 6 & 3 \\ 9 & 6 & 12 \end{bmatrix}} = 3 \times \underset{\mathbf{A}}{\begin{bmatrix} 5 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}}$$

2.3.3 Determinant

The determinant of a square matrix A of size m × m denoted as det (A) is a scalar calculated recursively as shown below:

1. If $m = 1$, $\det(\mathbf{A}) = a_{11}$
2. If $m > 1$, $\det(\mathbf{A}) = \sum (-1)^{i+j} \times a_{ij} \times \det(\mathbf{A}_{ij})$

Where $\mathbf{A}_{ij}$ is a matrix obtained from $\mathbf{A}$ by deleting the ith row and jth column.

Note

The determinant is defined only for a square matrix.

2.3.3 *Continued*

Example 2. 32

Figure 2.24 shows how we can calculate the determinant of a 2 × 2 matrix based on the determinant of a 1 × 1 matrix.

Figure 2.24 ***Calculating the determinant of a 2 × 2 matrix***

$$\det\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix} = (-1)^{1+1} \times 5 \times \det[\,4\,] + (-1)^{1+2} \times 2 \times \det[\,3\,] \longrightarrow 5 \times 4 - 2 \times 3 = 14$$

or

$$\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

Example 2. 33

Figure 2.25 shows the calculation of the determinant of a 3 × 3 matrix.

Figure 2.25 ***Calculating the determinant of a 3 × 3 matrix***

$$\det\begin{bmatrix} 5 & 2 & 1 \\ 3 & 0 & -4 \\ 2 & 1 & 6 \end{bmatrix} = (-1)^{1+1} \times 5 \times \det\begin{bmatrix} 0 & -4 \\ 1 & 6 \end{bmatrix} + (-1)^{1+2} \times 2 \times \det\begin{bmatrix} 3 & -4 \\ 2 & 6 \end{bmatrix} + (-1)^{1+3} \times 1 \times \det\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= (+1) \times 5 \times (+4) \quad + \quad (-1) \times 2 \times (24) \quad + \quad (+1) \times 1 \times (3) = -25$$

2.3.4 Inverses

Note

Multiplicative inverses are only defined for square matrices.

2.3.5 Residue Matrices

Cryptography uses residue matrices: matrices where all elements are in Z_n. A residue matrix has a multiplicative inverse if gcd (det(A), n) = 1.

Example 2. 34

Figure 2.26 ***A residue matrix and its multiplicative inverse***

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 7 & 2 \\ 1 & 4 & 7 & 2 \\ 6 & 3 & 9 & 17 \\ 13 & 5 & 4 & 16 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 15 & 21 & 0 & 15 \\ 23 & 9 & 0 & 22 \\ 15 & 16 & 18 & 3 \\ 24 & 7 & 15 & 3 \end{bmatrix}$$

$$\det(\mathbf{A}) = 21 \qquad \det(\mathbf{A}^{-1}) = 5$$

2-4 LINEAR CONGRUENCE

Cryptography often involves solving an equation or a set of equations of one or more variables with coefficient in Z_n. This section shows how to solve equations when the power of each variable is 1 (linear equation).

Topics discussed in this section:

2.4.1 Single-Variable Linear Equations
2.4.2 Set of Linear Equations

2.4.1 Single-Variable Linear Equations

Equations of the form $ax \equiv b \pmod{n}$ might have no solution or a limited number of solutions.

Assume that the gcd $(a, n) = d$.

If $d \nmid b$, there is no solution.

If $d \mid b$, there are d solutions.

Example 2.35

Solve the equation 10 $x \equiv 2 \pmod{15}$.

Solution

First we find the gcd (10 and 15) = 5. Since 5 does not divide 2, we have no solution.

Example 2.36

Solve the equation 14 $x \equiv 12 \pmod{18}$.

Solution

$$14x \equiv 12 \pmod{18} \rightarrow \quad 7x \equiv 6 \pmod 9 \quad \rightarrow \quad x \equiv 6\ (7^{-1}) \pmod 9$$

$$x_0 = (6 \times 7^{-1}) \bmod 9 = (6 \times 4) \pmod 9 = 6$$

$$x_1 = x_0 + 1 \times (18/2) = 15$$

2.4.1 Continued

Example 2.37

Solve the equation $3x + 4 \equiv 6 \pmod{13}$.

Solution

First we change the equation to the form $ax \equiv b \pmod{n}$. We add −4 (the additive inverse of 4) to both sides, which give $3x \equiv 2 \pmod{13}$. Because gcd (3, 13) = 1, the equation has only one solution, which is $x_0 = (2 \times 3^{-1}) \bmod 13 = 18 \bmod 13 = 5$. We can see that the answer satisfies the original equation: $3 \times 5 + 4 \equiv 6 \pmod{13}$.

2.4.2 Single-Variable Linear Equations

We can also solve a set of linear equations with the same modulus if the matrix formed from the coefficients of the variables is invertible.

Figure 2.27 ***Set of linear equations***

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\equiv b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\equiv b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &\equiv b_n \end{aligned}$$

a. Equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

b. Interpretation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

c. Solution

2.4.2 Continued

Example 2.38

Solve the set of following three equations:

$$3x + 5y + 7z \equiv 3 \pmod{16}$$
$$x + 4y + 13z \equiv 5 \pmod{16}$$
$$2x + 7y + 3z \equiv 4 \pmod{16}$$

Solution

The result is $x \equiv 15 \pmod{16}$, $y \equiv 4 \pmod{16}$, and $z \equiv 14 \pmod{16}$. We can check the answer by inserting these values into the equations.