Lower Bounds

Lower bound: an estimate on a minimum amount of work needed to solve a given problem

Examples:

- number of comparisons needed to find the largest element in a set of *n* numbers
- number of comparisons needed to sort an array of size *n*
- number of comparisons necessary for searching in a sorted array
- number of multiplications needed to multiply two *n*-by-*n* matrices

Lower Bounds (cont.)

- Lower bound can be
  - an exact count
  - an efficiency class (Ω)
- ***Tight*** lower bound: there exists an algorithm with the same efficiency as the lower bound

Problem	Lower bound	Tightness
sorting	$\Omega(n\log n)$	yes
searching in a sorted array	$\Omega(\log n)$	yes
element uniqueness	$\Omega(n\log n)$	yes
n-digit integer multiplication	$\Omega(n)$	unknown
multiplication of n-by-n matrices	$\Omega(n^2)$	unknown

Methods for Establishing Lower Bounds

- trivial lower bounds
- information-theoretic arguments (decision trees)
- adversary arguments
- problem reduction

Trivial Lower Bounds

Trivial lower bounds: based on counting the number of items that must be processed in input and generated as output

Examples

- finding max element
- polynomial evaluation
- sorting
- element uniqueness
- Hamiltonian circuit existence

Conclusions

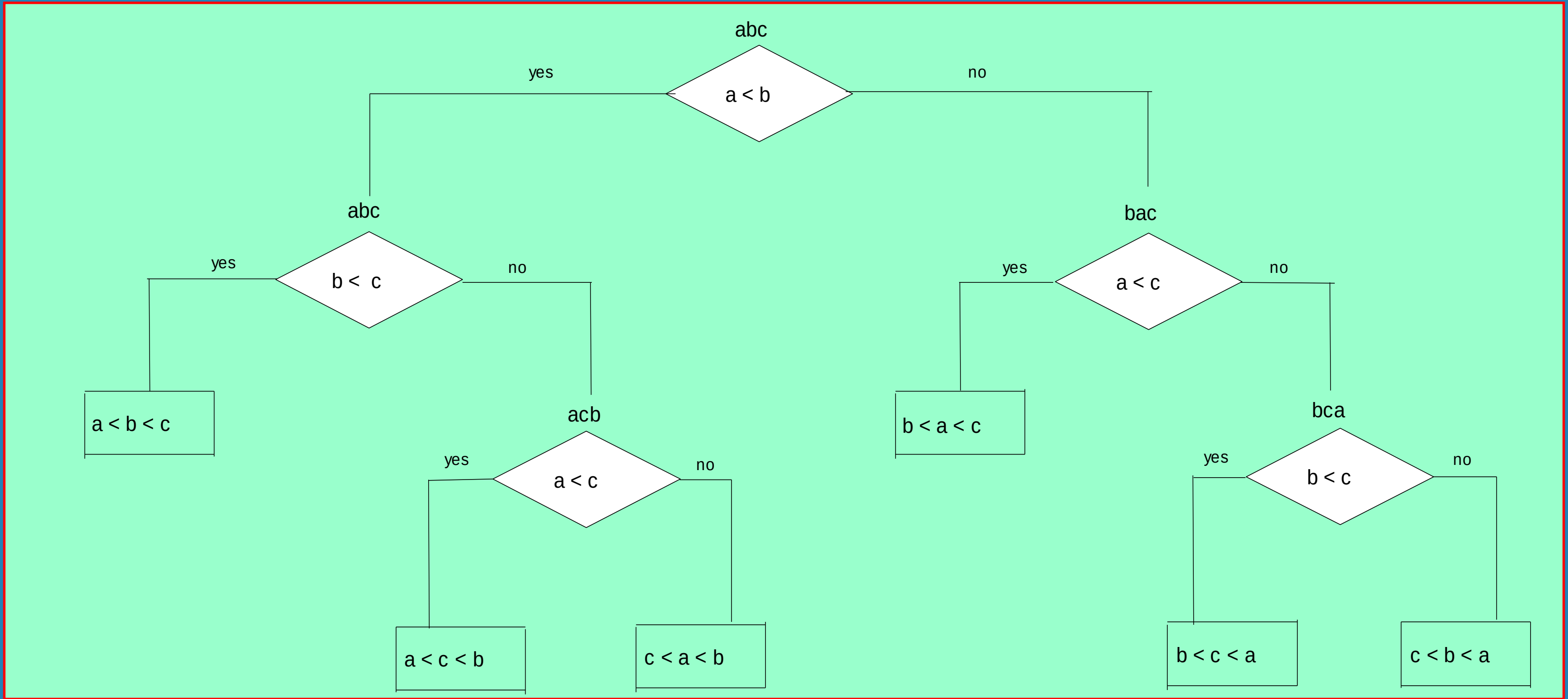
- may and may not be useful
- be careful in deciding how many elements must be processed

Decision Trees

Decision tree **— a convenient model of algorithms involving comparisons in which:**

- **internal nodes represent comparisons**
- **leaves represent outcomes**

Decision tree for 3-element insertion sort



Decision Trees and Sorting Algorithms

- Any comparison-based sorting algorithm can be represented by a decision tree
- Number of leaves (outcomes) $\geq n!$
- Height of binary tree with $n!$ leaves $\geq \lceil \log_2 n! \rceil$
- Minimum number of comparisons in the worst case $\geq \lceil \log_2 n! \rceil$ for any comparison-based sorting algorithm
- $\lceil \log_2 n! \rceil \approx n \log_2 n$
- This lower bound is tight (mergesort)

Adversary Arguments

Adversary argument: a method of proving a lower bound by playing role of adversary that makes algorithm work the hardest by adjusting input

Example 1: "Guessing" a number between 1 and *n* with yes/no questions

Adversary: Puts the number in a larger of the two subsets generated by last question

Example 2: Merging two sorted lists of size *n*

$a_1 < a_2 < \ldots < a_n$ **and** $b_1 < b_2 < \ldots < b_n$

Adversary: $a_i < b_j$ **iff** $i < j$

Output $b_1 < a_1 < b_2 < a_2 < \ldots < b_n < a_n$ **requires 2*n*-1 comparisons of adjacent elements**

Lower Bounds by Problem Reduction

Idea: If problem *P* is at least as hard as problem *Q*, then a lower bound for *Q* is also a lower bound for *P*.

Hence, find problem *Q* with a known lower bound that can be reduced to problem *P* in question.

Example: *P* is finding MST for *n* points in Cartesian plane
***Q* is element uniqueness problem (known to be in $\Omega(n\log n)$)**

Classifying Problem Complexity

Is the problem *tractable*, i.e., is there a polynomial-time (O(*p*(*n*)) algorithm that solves it?

Possible answers:

- **yes (give examples)**
- **no**
  - **because it's been proved that no algorithm exists at all (e.g., Turing's *halting problem*)**
  - **because it's been be proved that any algorithm takes exponential time**
- **unknown**

Problem Types: Optimization and Decision

- ***Optimization problem***: find a solution that maximizes or minimizes some objective function

- ***Decision problem***: answer yes/no to a question

Many problems have decision and optimization versions.

E.g.: traveling salesman problem

- ***optimization***: find Hamiltonian cycle of minimum length
- ***decision***: find Hamiltonian cycle of length ≤ m

Decision problems are more convenient for formal investigation of their complexity.

Class *P*

P: the class of decision problems that are solvable in O(*p*(*n*)) time, where *p*(*n*) is a polynomial of problem's input size *n*

Examples:

- searching
- element uniqueness
- graph connectivity
- graph acyclicity
- primality testing (finally proved in 2002)

Class *NP*

NP (***nondeterministic polynomial***): **class of decision problems whose proposed solutions can be verified in polynomial time = solvable by a *nondeterministic polynomial algorithm***

A *nondeterministic polynomial algorithm* is an abstract two-stage procedure that:

- **generates a random string purported to solve the problem**
- **checks whether this solution is correct in polynomial time**

By definition, it solves the problem if it's capable of generating and verifying a solution on one of its tries

Why this definition?

- **led to development of the rich theory called "computational complexity"**

Example: CNF satisfiability

Problem: Is a boolean expression in its conjunctive normal form (CNF) satisfiable, i.e., are there values of its variables that makes it true?

This problem is in *NP*. Nondeterministic algorithm:

- **Guess truth assignment**
- **Substitute the values into the CNF formula to see if it evaluates to true**

Example: (A | ¬B | ¬C) & (A | B) & (¬B | ¬D | E) & (¬D | ¬E)

Truth assignments:

<u>A B C D E</u>

0 0 0 0 0

. . .

1 1 1 1 1

Checking phase: O(*n*)

What problems are in *NP*?

- Hamiltonian circuit existence
- Partition problem: Is it possible to partition a set of *n* integers into two disjoint subsets with the same sum?
- Decision versions of TSP, knapsack problem, graph coloring, and many other combinatorial optimization problems. (Few exceptions include: MST, shortest paths)

- All the problems in *P* can also be solved in this manner (no guessing is necessary), so we have:

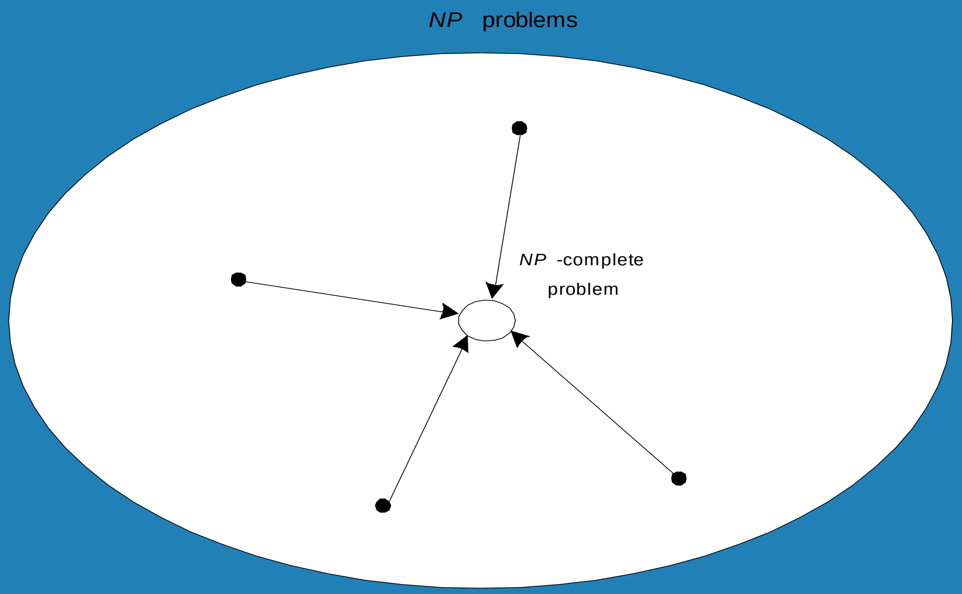
$$P \subseteq NP$$

- Big question: $P = NP$?

NP-Complete Problems

A decision problem *D* is ***NP*-complete** if it's as hard as any problem in *NP*, i.e.,

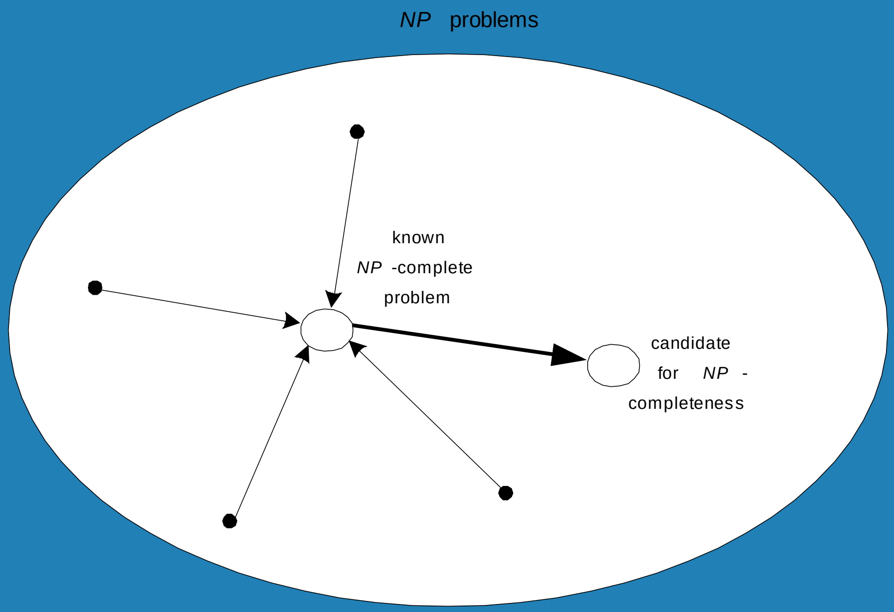
- *D* is in *NP*
- every problem in *NP* is polynomial-time reducible to *D*



Cook's theorem (1971): CNF-sat is *NP*-complete

NP-Complete Problems (cont.)

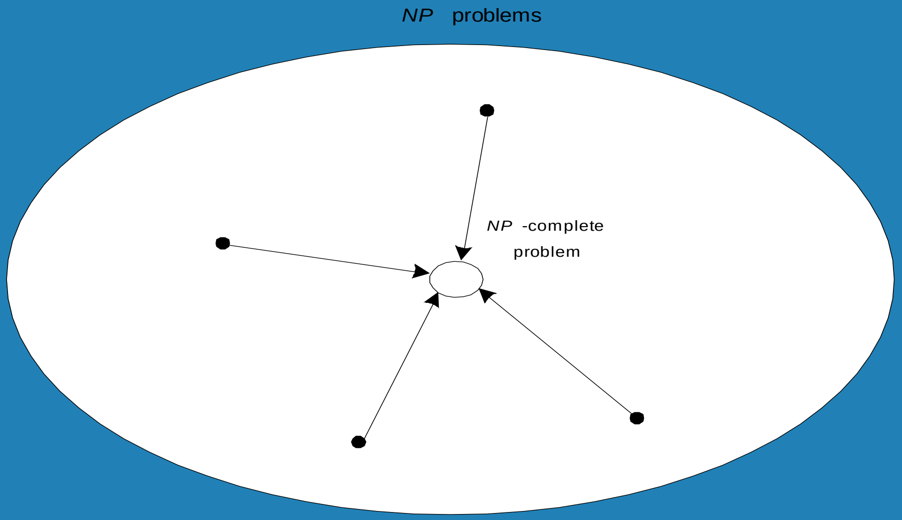
Other *NP*-complete problems obtained through polynomial-time reductions from a known *NP*-complete problem



Examples: TSP, knapsack, partition, graph-coloring and hundreds of other problems of combinatorial nature

$P = NP$? Dilemma Revisited

- $P = NP$ would imply that every problem in NP, including all NP-complete problems, could be solved in polynomial time
- If a polynomial-time algorithm for just one NP-complete problem is discovered, then every problem in NP can be solved in polynomial time, i.e., $P = NP$



- Most but not all researchers believe that $P \neq NP$, i.e. P is a proper subset of NP