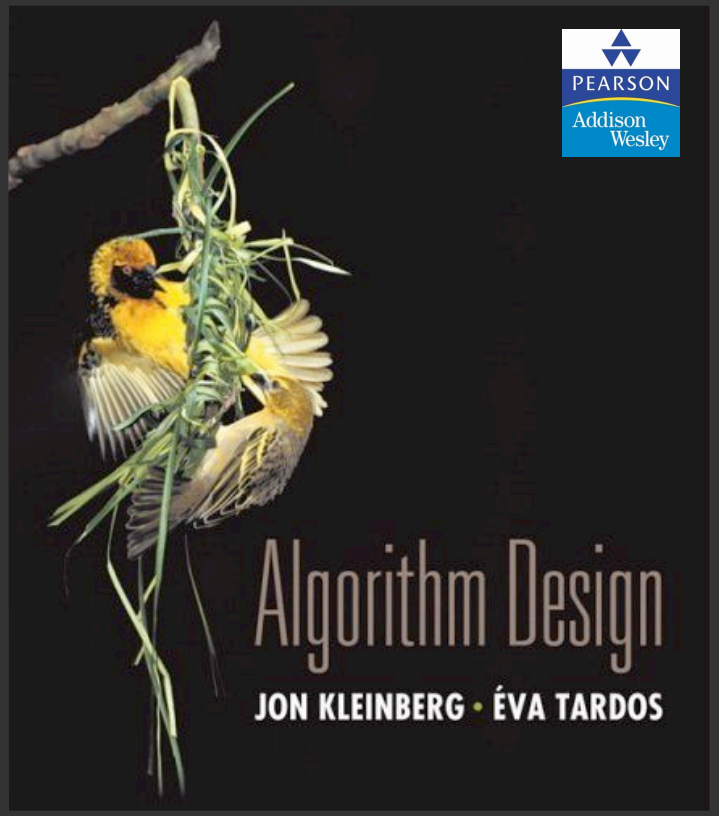


5. Divide And Conquer I

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on Oct 2, 2013 9:51 AM

Divide-and-conquer paradigm

Divide-and-conquer.

- Divide up problem into several subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems into overall solution.

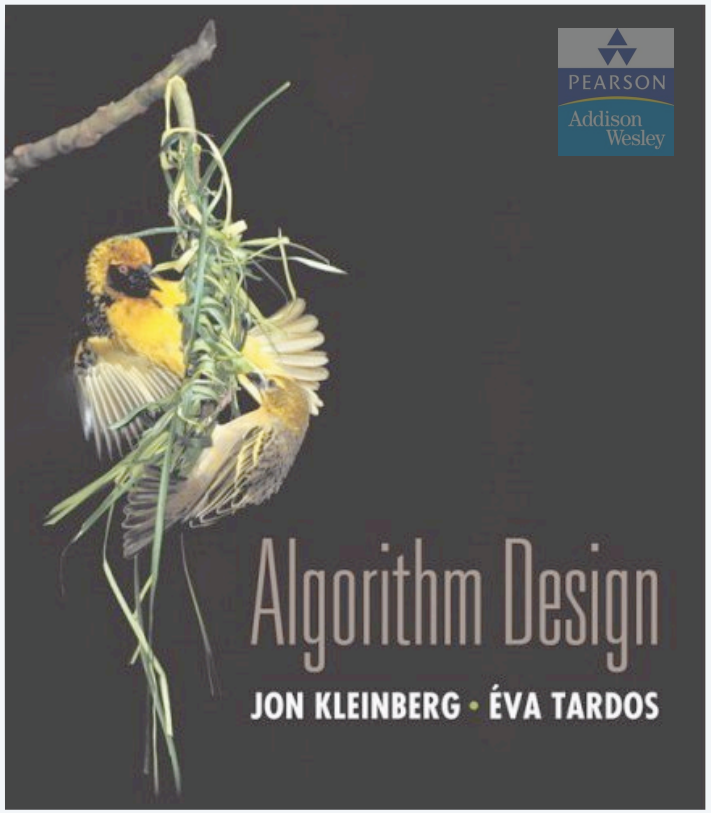
Most common usage.

- Divide problem of size n into two subproblems of size $n/2$ in linear time.
- Solve two subproblems recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $\Theta(n \log n)$.

attributed to Julius Caesar



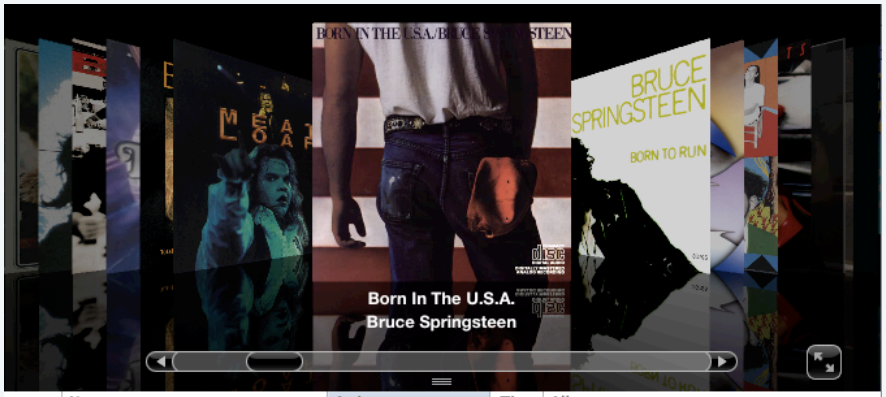
SECTION 5.1

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Sorting problem

Problem. Given a list of n elements from a totally-ordered universe, rearrange them in ascending order.



	Name	Artist	Time	Album
12	Let It Be	The Beatles	4:03	Let It Be
13	Take My Breath Away	BERLIN	4:13	Top Gun – Soundtrack
14	Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	Dancing With Myself	Billy Idol	4:43	Don't Stop
16	Rebel Yell	Billy Idol	4:49	Rebel Yell
17	Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
19	The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
20	Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	Hurricane	Bob Dylan	8:32	Desire
25	The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	Beds Of Roses	Bon Jovi	6:35	Cross Road
28	Runaway	Bon Jovi	3:53	Cross Road
29	Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	Thunder Road	Bruce Springsteen	4:51	Born To Run
36	Born To Run	Bruce Springsteen	4:30	Born To Run
37	Jungleland	Bruce Springsteen	9:34	Born To Run

Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

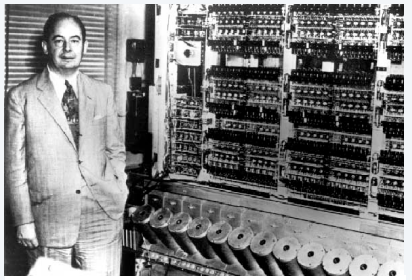
- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

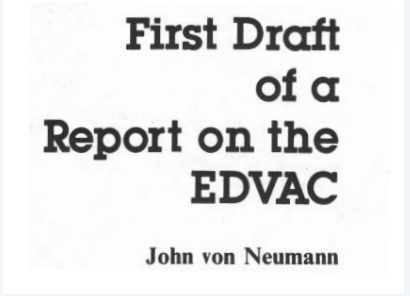
Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.





input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

sort right half

A	G	L	O	R	H	I	M	S	T

merge results

A	G	H	I	L	M	O	R	S	T

Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.

Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace $\leq$ with $=$.

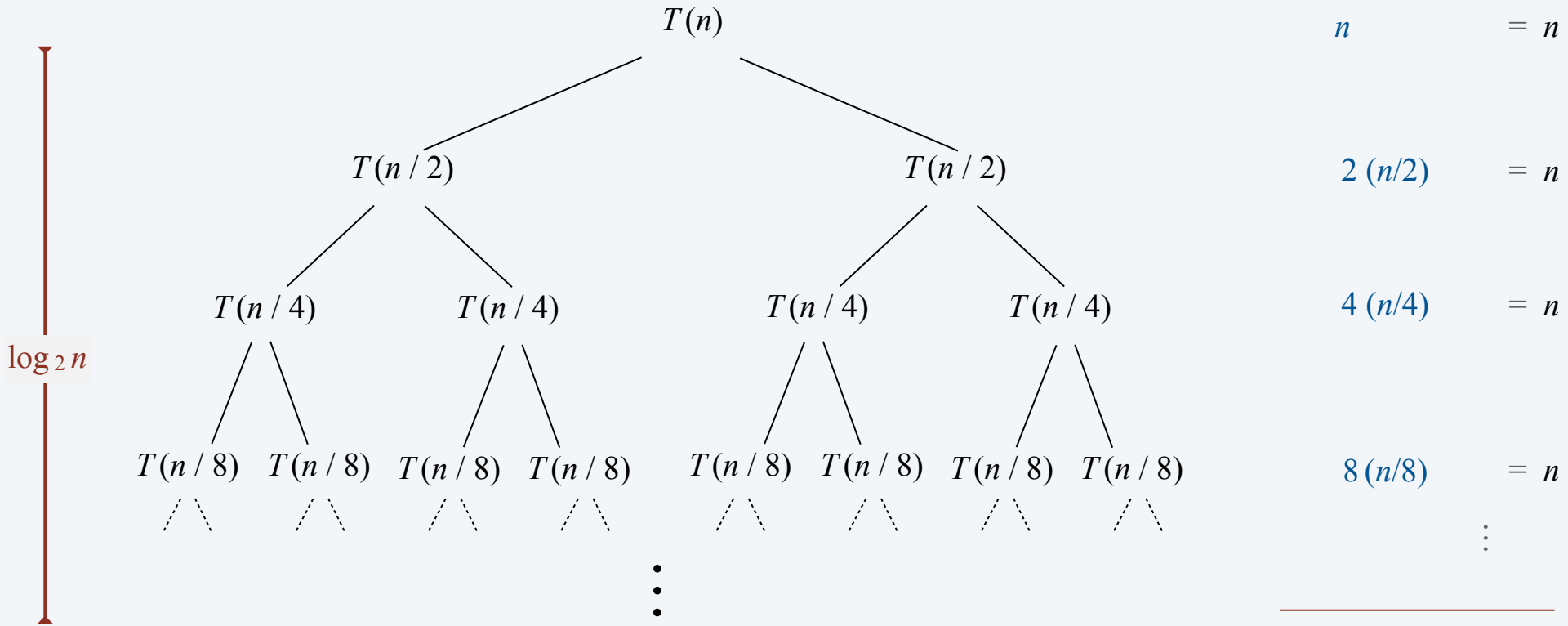
Divide-and-conquer recurrence: proof by recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 1.



Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad \blacksquare \end{aligned}$$

Analysis of mergesort recurrence

Claim. If $T(n)$ satisfies the following recurrence, then $T(n) \leq n \lceil \log_2 n \rceil$.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

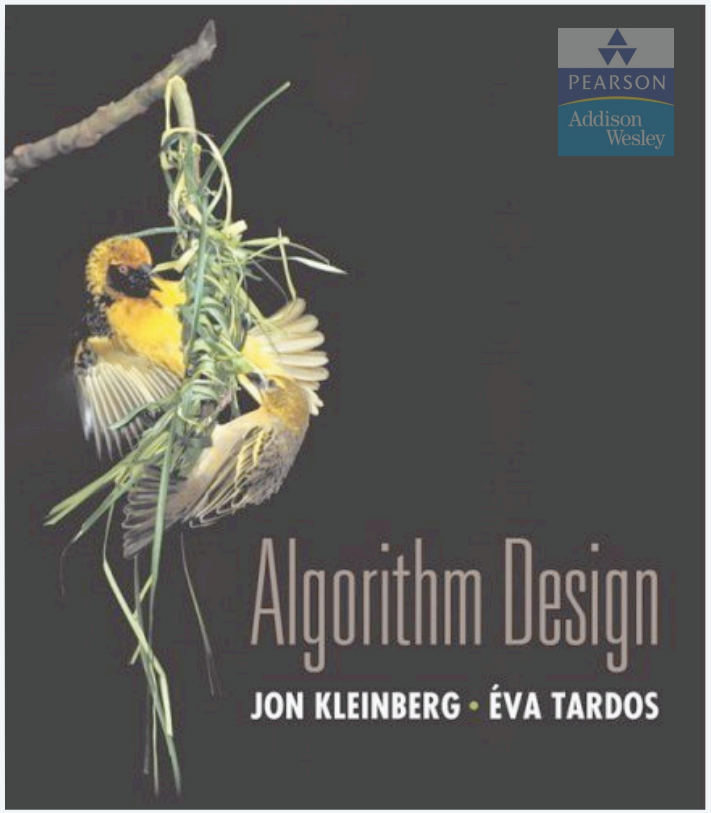
Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.
- Induction step: assume true for $1, 2, \ldots, n-1$.

$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + n \\ &\leq n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\leq n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \\ &\leq n (\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil. \quad \blacksquare \end{aligned}$$

$\log_2 n_2 \leq \lceil \log_2 n \rceil - 1$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\leq \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \end{aligned}$$



SECTION 5.3

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, \ldots, n$.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

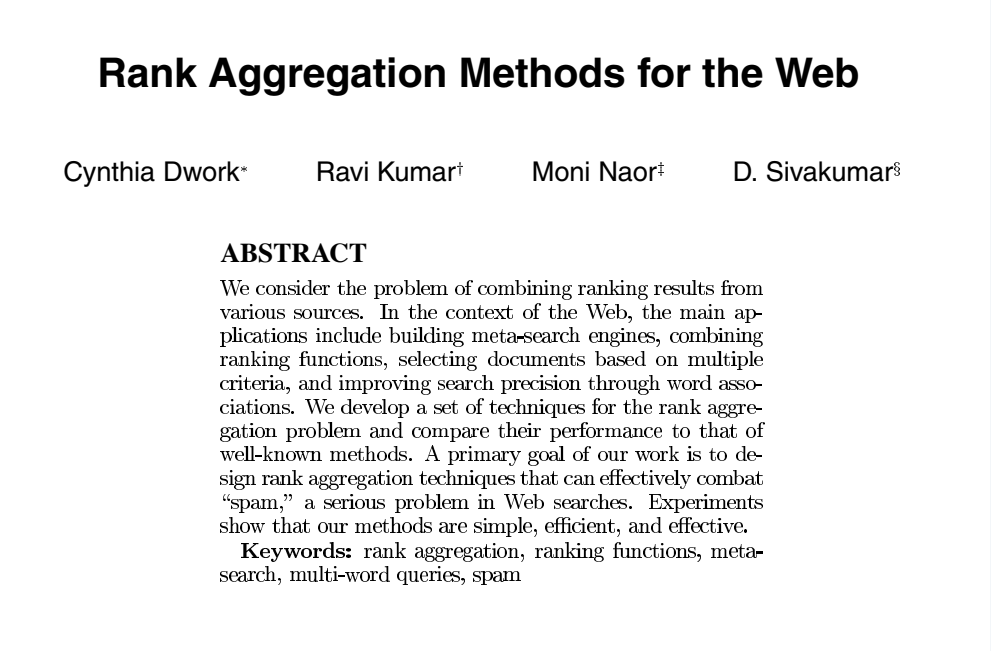
	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's tau distance).



Rank Aggregation Methods for the Web

Cynthia Dwork* Ravi Kumar† Moni Naor‡ D. Sivakumar§

ABSTRACT

We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat "spam," a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10

5-4

count inversions in right half B

2	6	9	3	7

6-3 9-3 9-7

count inversions (a, b) with a ∈ A and b ∈ B

1	5	4	8	10

2	6	9	3	7

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output 1 + 3 + 13 = 17

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element $b \in B$,
  - binary search in A to find how elements in A are greater than b.

list A

7	10	18	3	14

list B

17	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	17	23

binary search to count inversions (a, b) with a ∈ A and b ∈ B

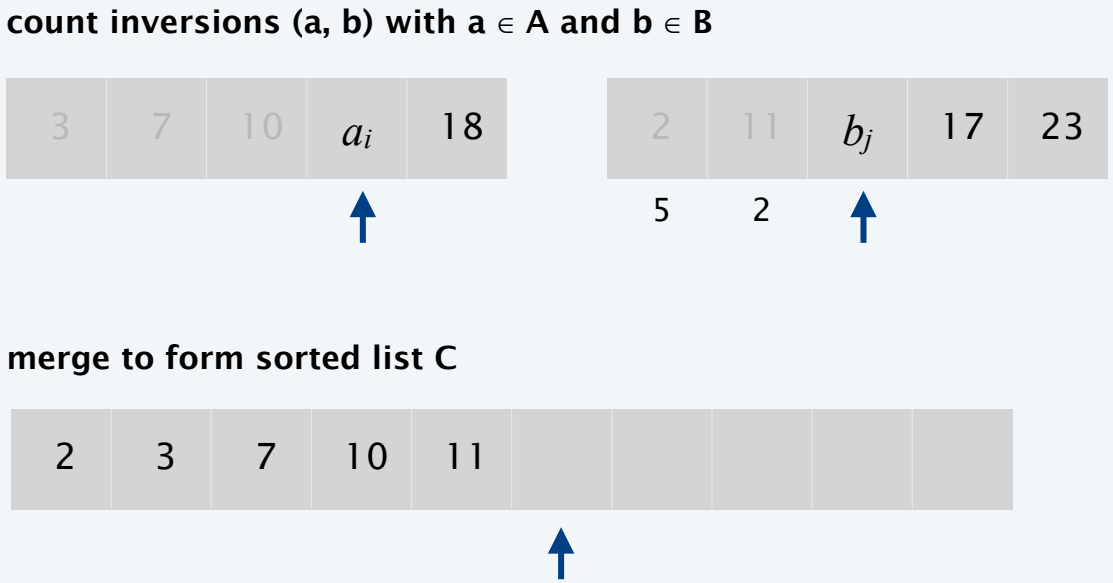
3	7	10	14	18

2	11	16	17	23
5	2	1	1	0

Counting inversions: how to combine two subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Counting inversions: divide-and-conquer algorithm implementation

Input. List L.

Output. Number of inversions in L and sorted list of elements L'.

SORT-AND-COUNT (L)

IF list L has one element

    RETURN $(0, L)$.

DIVIDE the list into two halves A and B.

$(r_A , A) \leftarrow$ SORT-AND-COUNT(A).

$(r_B , B) \leftarrow$ SORT-AND-COUNT(B).

$(r_{AB} , L') \leftarrow$ MERGE-AND-COUNT(A, B).

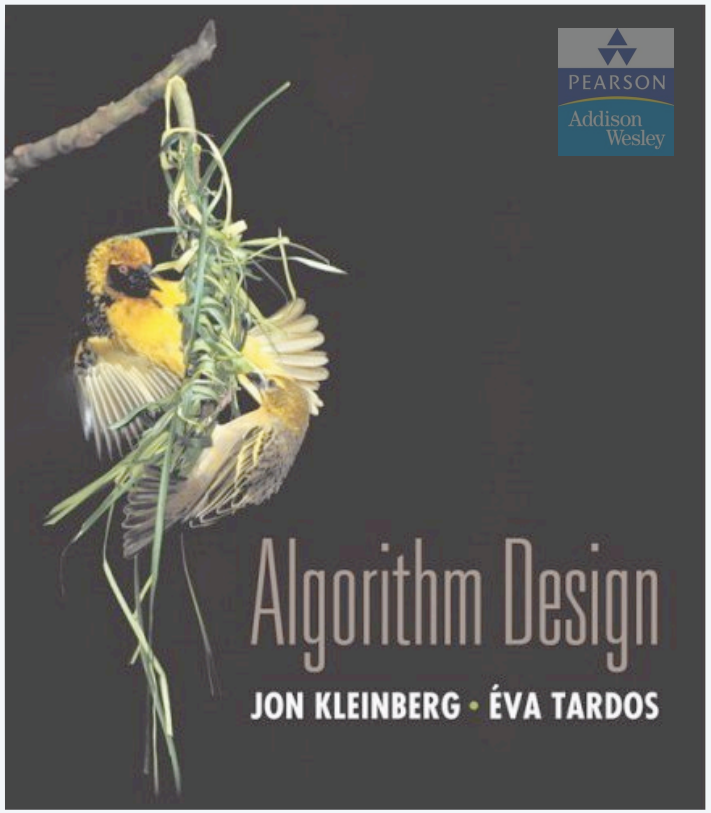
RETURN $(r_A + r_B + r_{AB} , L')$.

Counting inversions: divide-and-conquer algorithm analysis

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$



SECTION 5.4

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

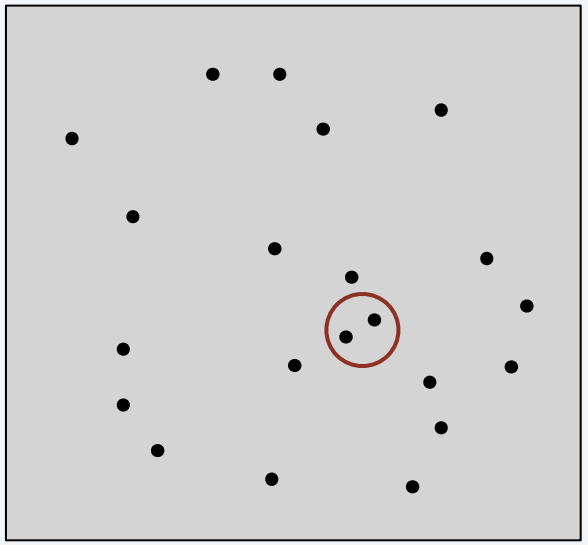
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems



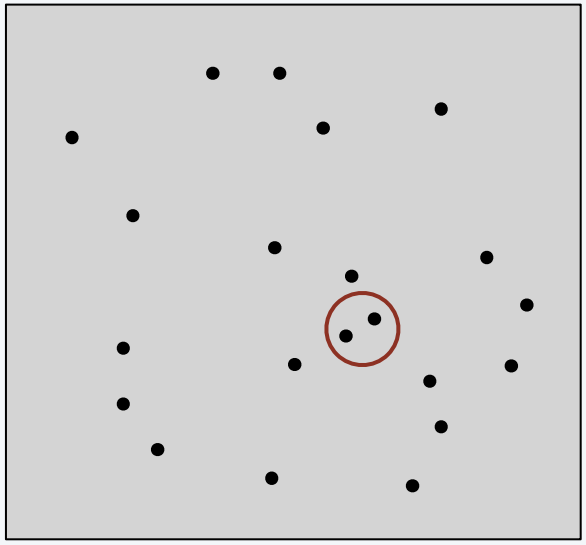
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1d version. Easy $O(n \log n)$ algorithm if points are on a line.

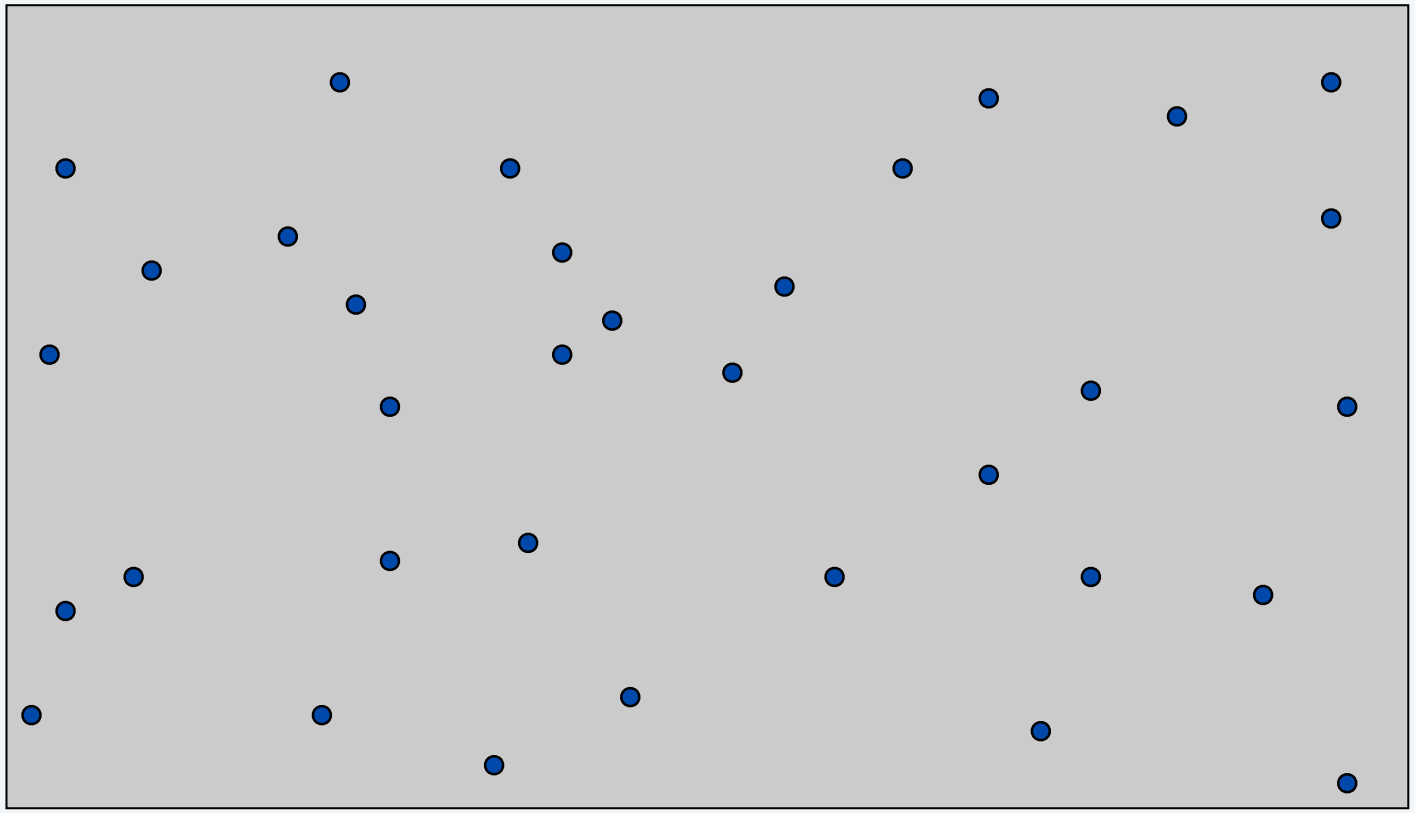
Nondegeneracy assumption. No two points have the same x-coordinate.



Closest pair of points: first attempt

Sorting solution.

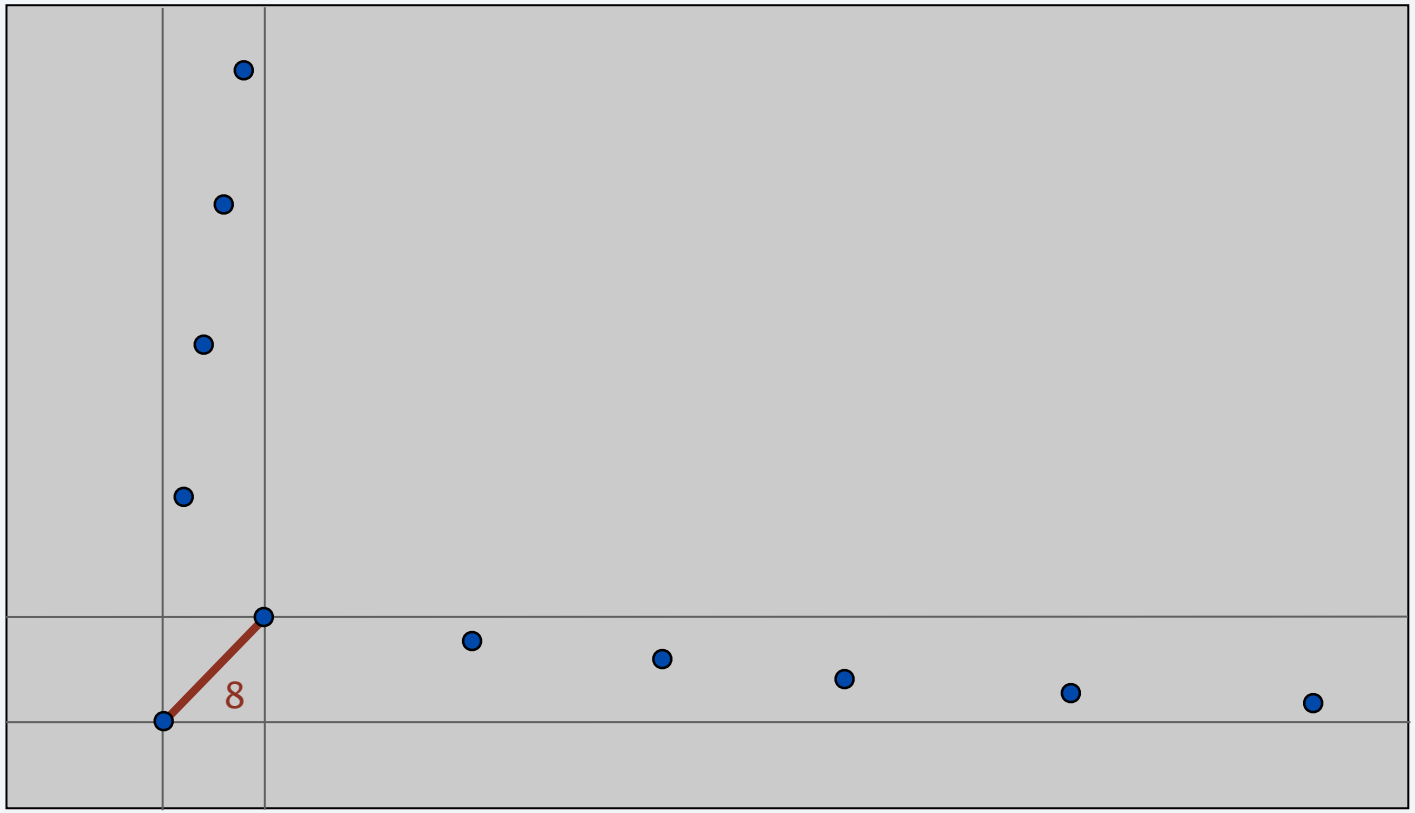
- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: first attempt

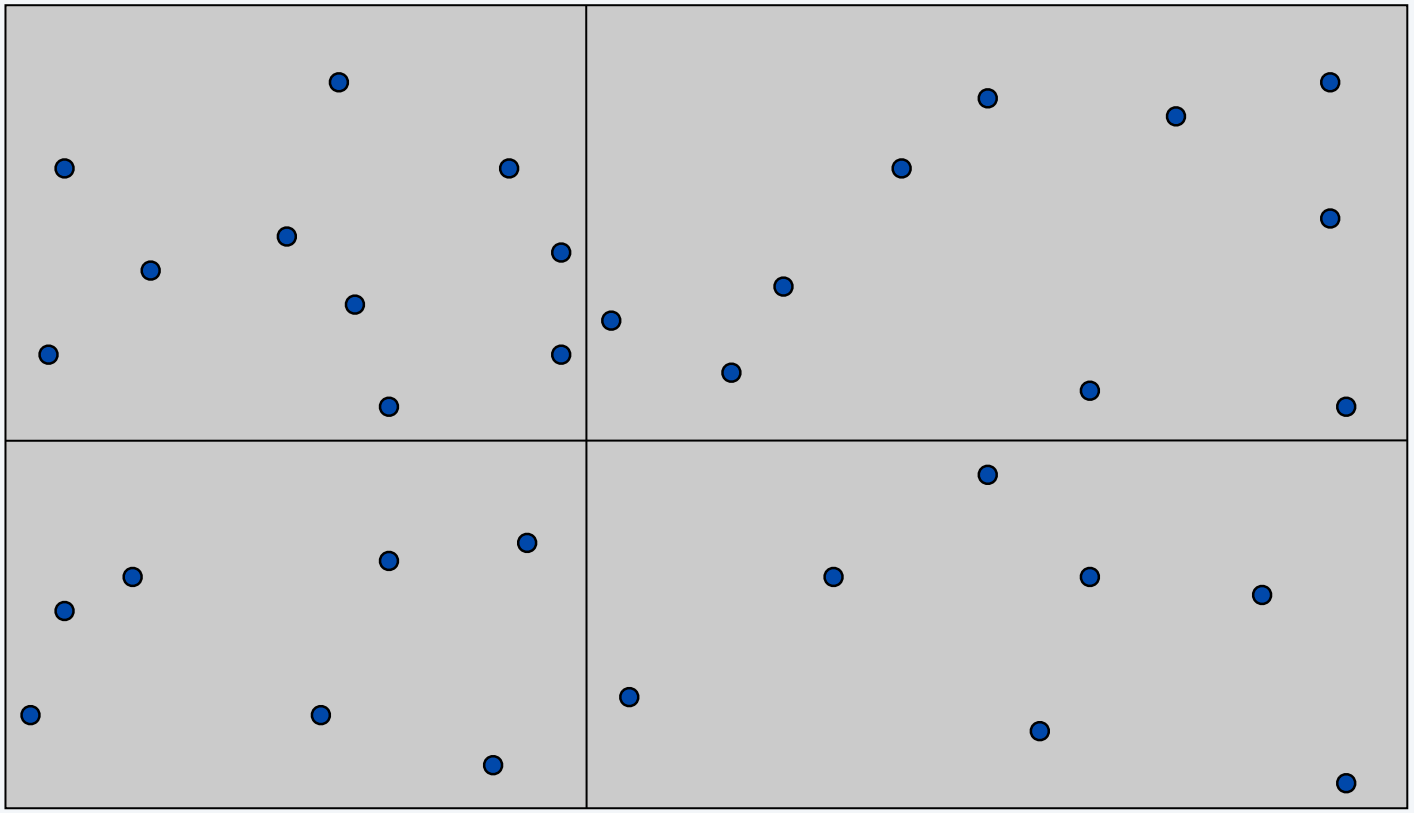
Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: second attempt

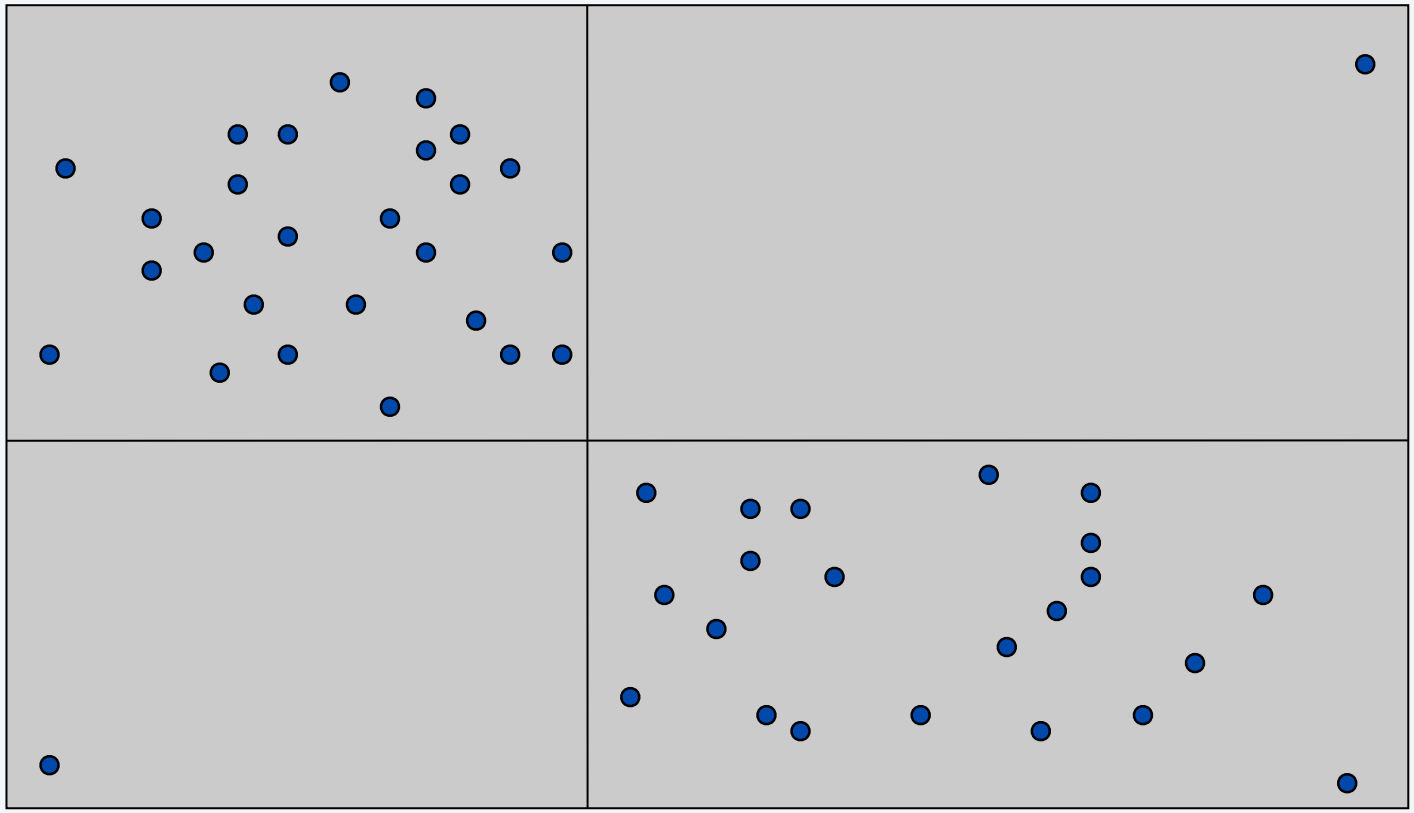
Divide. Subdivide region into 4 quadrants.



Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

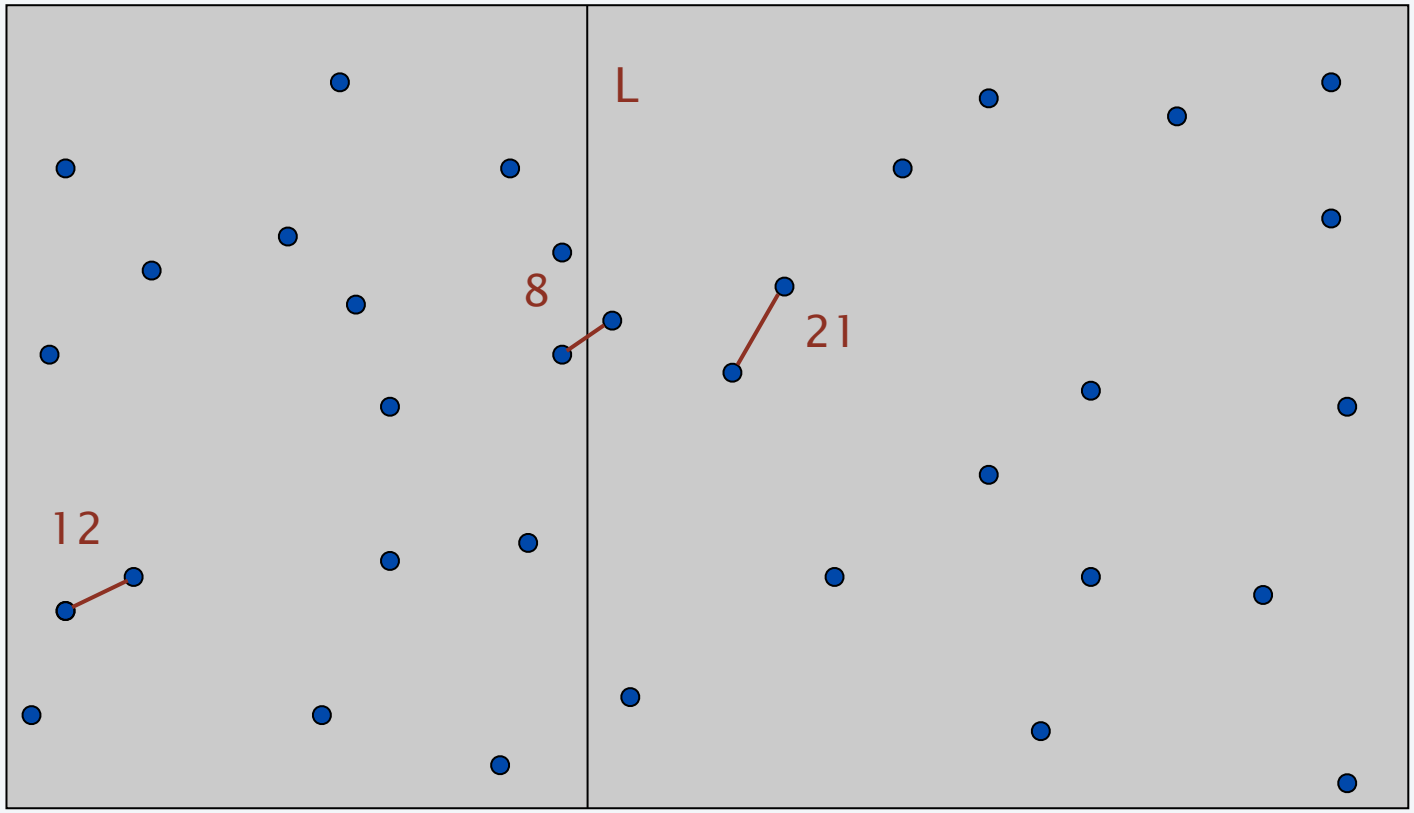
Obstacle. Impossible to ensure $n/4$ points in each piece.



Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

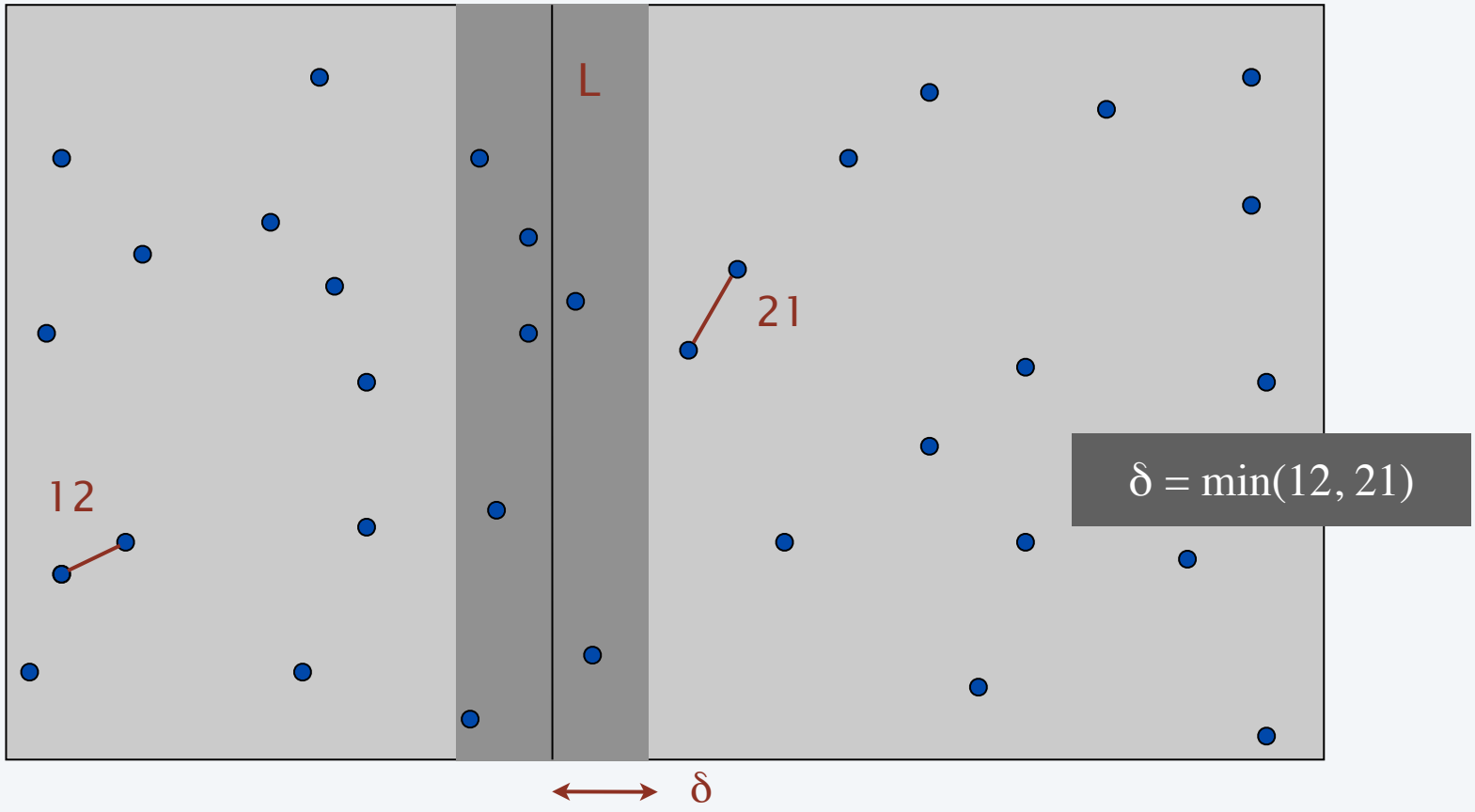
seems like $\Theta(N^2)$



How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.

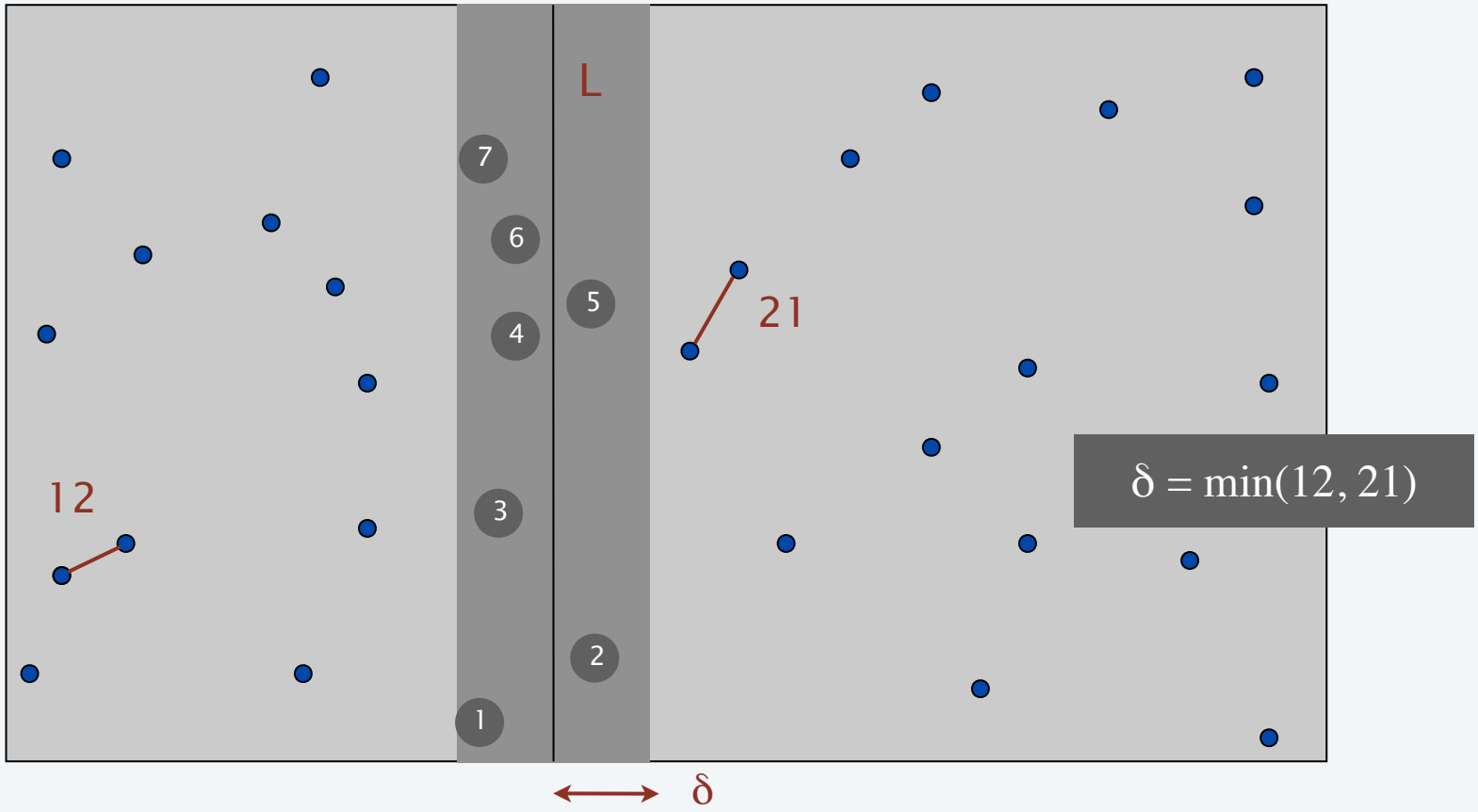


How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Only check distances of those within 11 positions in sorted list!

why 11?



How to find closest pair with one point in each side?

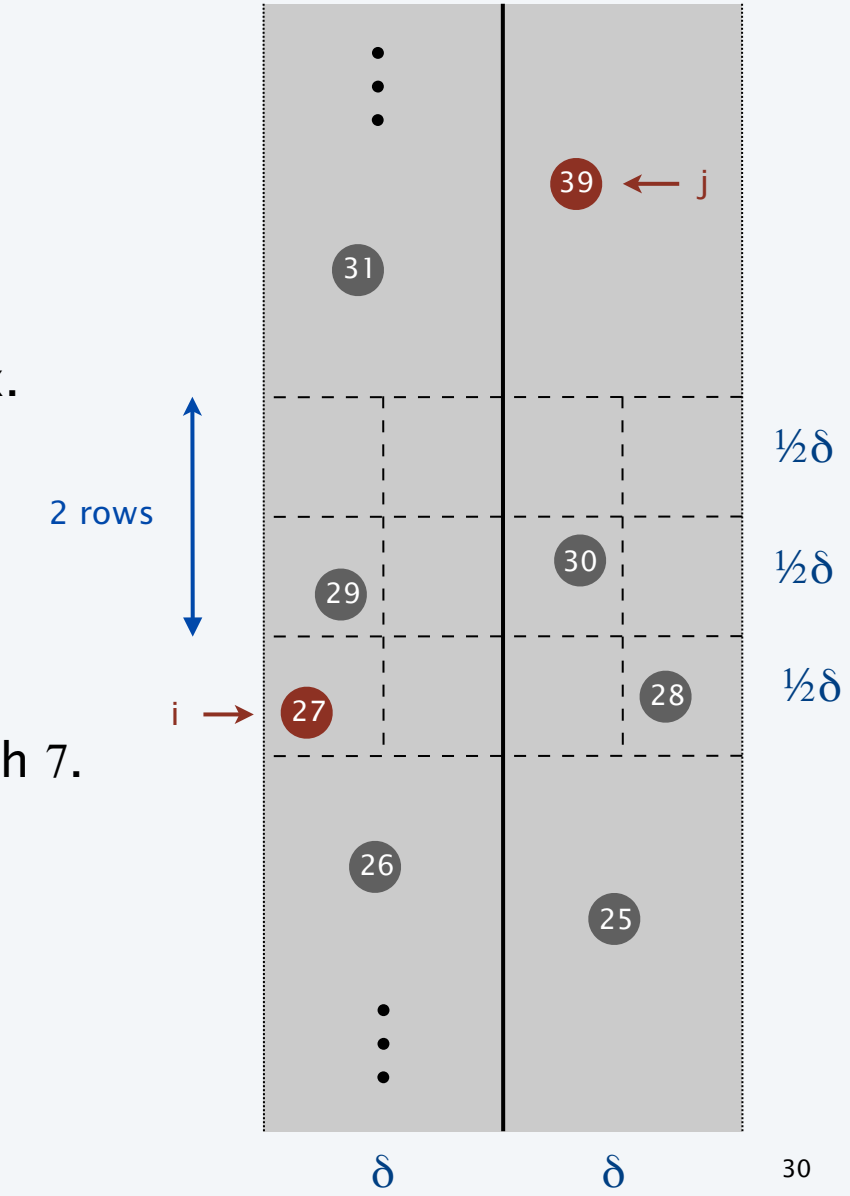
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Claim remains true if we replace 12 with 7.



Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR $(p_1, p_2, \ldots, p_n)$

Compute separation line L such that half the points are on each side of the line. ← $O(n \log n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR (points in left half).

$\delta_2 \leftarrow$ CLOSEST-PAIR (points in right half). ← $2\,T(n / 2)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$.

Delete all points further than δ from line L. ← $O(n)$

Sort remaining points by y-coordinate. ← $O(n \log n)$

Scan points in y-order and compare distance between each point and next 11 neighbors. If any of these distances is less than δ, update δ. ← $O(n)$

RETURN δ.

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n \log n) & \text{otherwise} \end{cases}$$

$(x_1 - x_2)^2 + (y_1 - y_2)^2$

Lower bound. In quadratic decision tree model, any algorithm for closest pair (even in 1D) requires $\Omega(n \log n)$ quadratic tests.

Improved closest pair algorithm

Q. How to improve to $O(n \log n)$?

A. Yes. Don't sort points in strip from scratch each time.

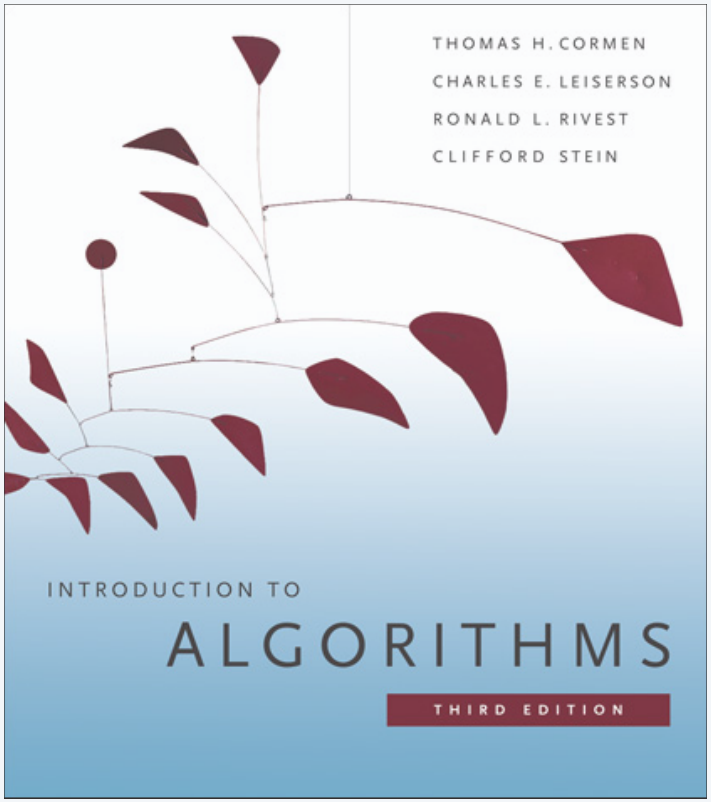
- Each recursive returns two lists: all points sorted by x-coordinate, and all points sorted by y-coordinate.
- Sort by merging two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Note. See SECTION 13.7 for a randomized $O(n)$ time algorithm.

not subject to lower bound
since it uses the floor function



CHAPTER 7

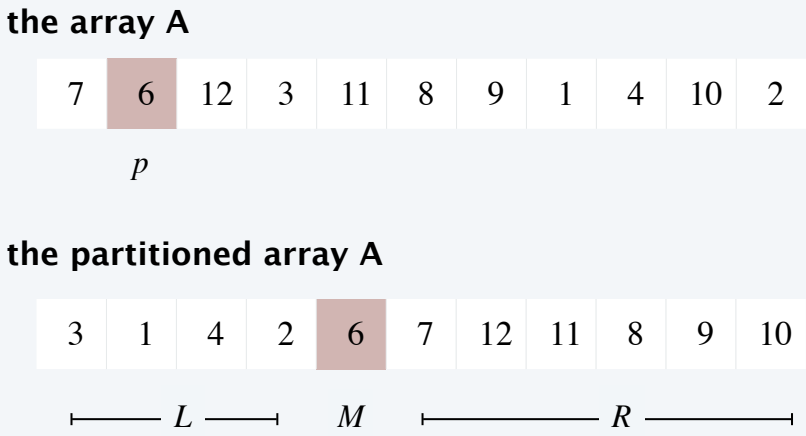
5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Randomized quicksort

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.



Recur in both left and right subarrays.

RANDOMIZED-QUICKSORT (A)

IF list A has zero or one element

  RETURN.

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, a_i).

RANDOMIZED-QUICKSORT(L).

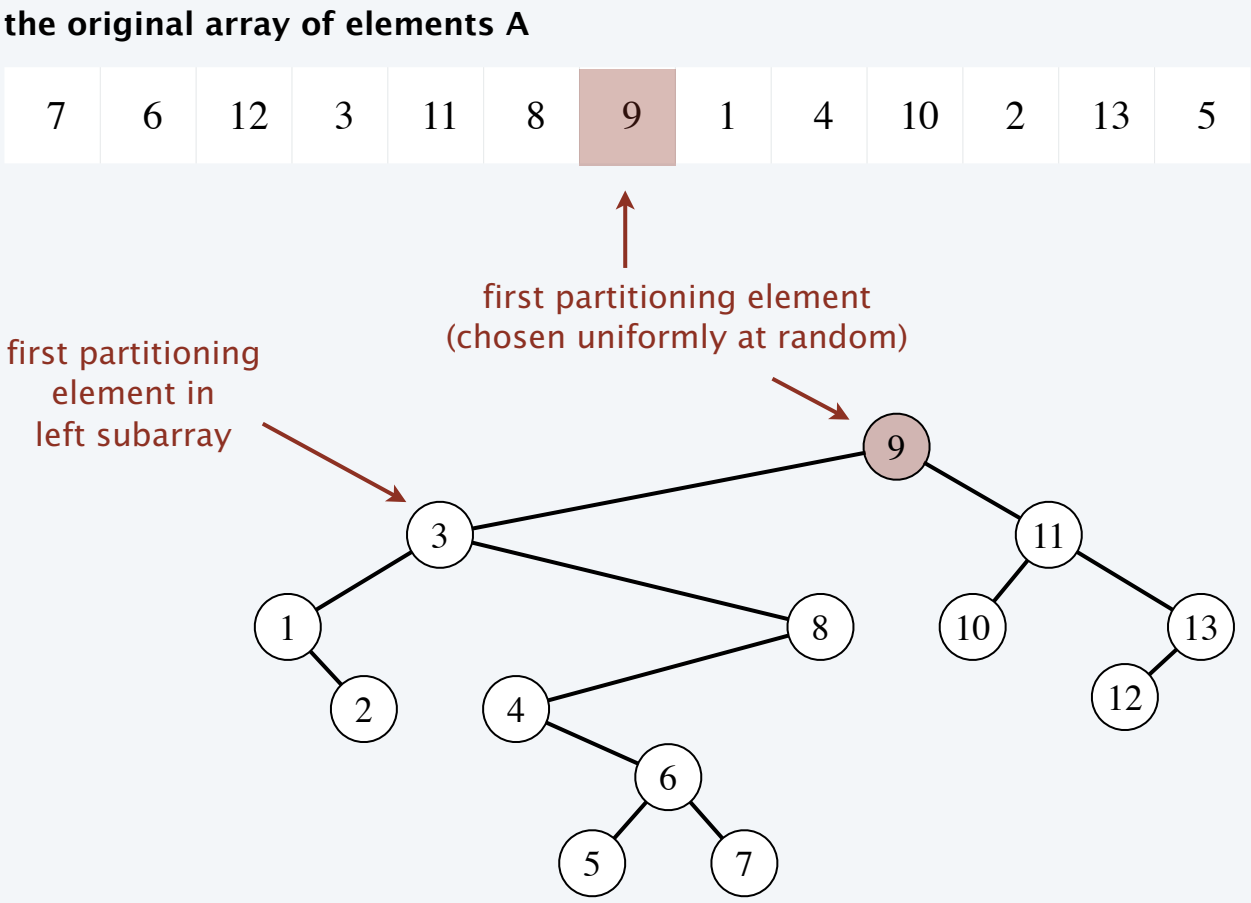
RANDOMIZED-QUICKSORT(R).

3-way partitioning can be done in-place (using n–1 compares)

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.



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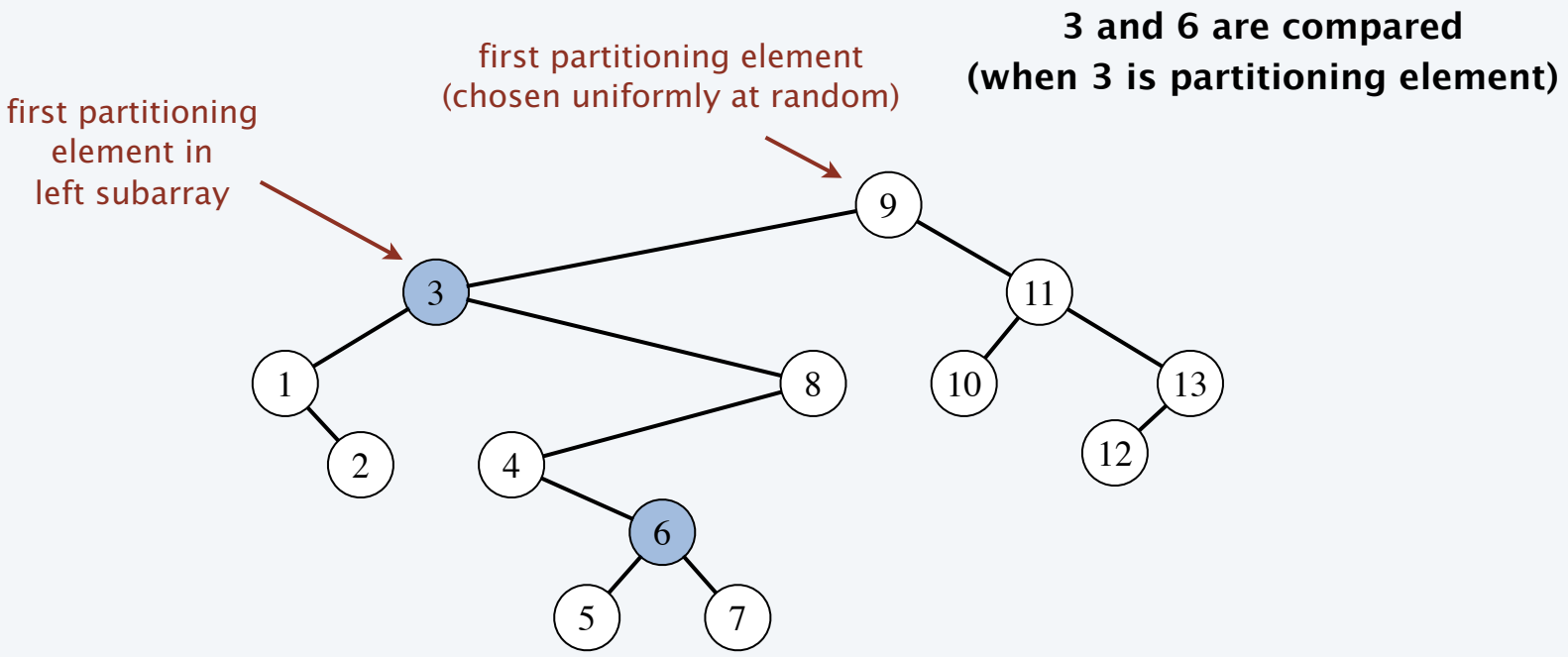
- An element is compared with only its ancestors and descendants.



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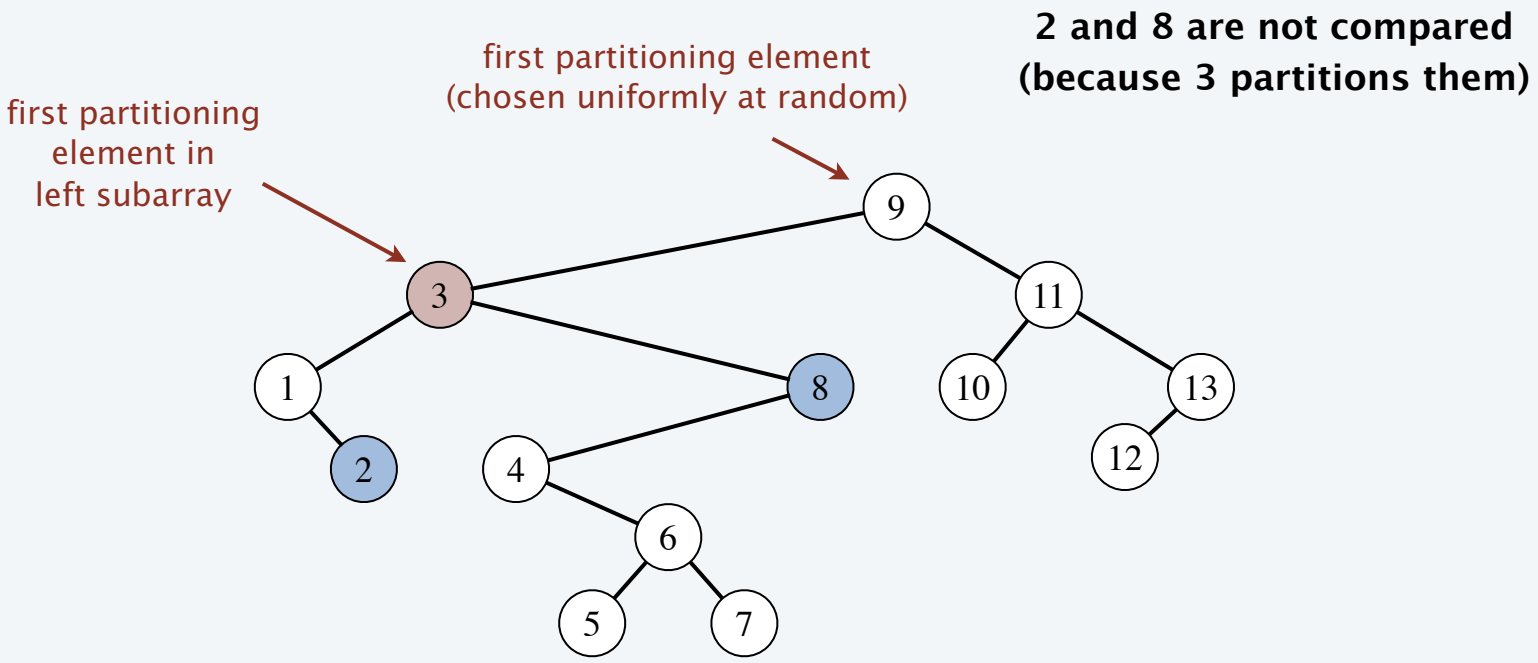

Analysis of randomized quicksort

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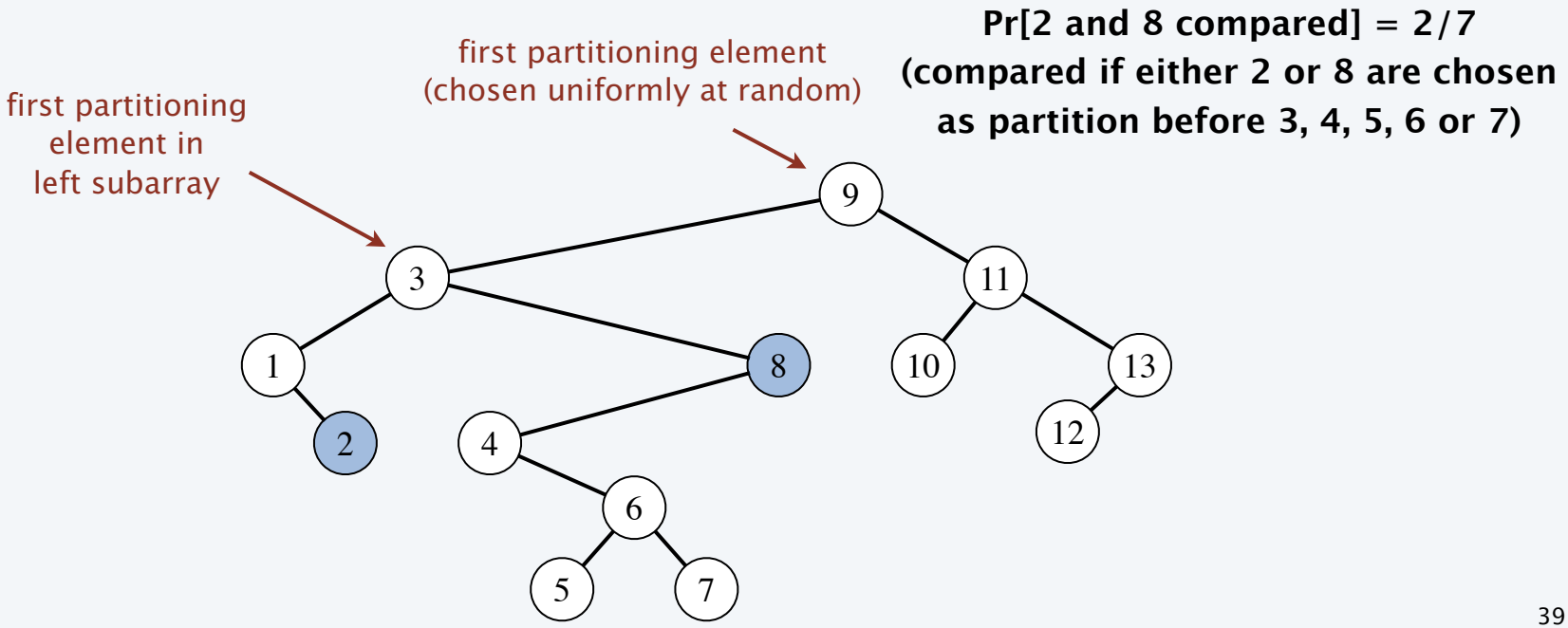
- An element is compared with only its ancestors and descendants.
- **Pr** [a_i and a_j are compared] $= 2 \ / \ |j - i + 1|$.



Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.
- **Pr** [a_i and a_j are compared] $= 2 \ / \ |j - i + 1|$.
- Expected number of compares $=$

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{2}{j-i+1} = 2 \sum_{i=1}^{N} \sum_{j=2}^{N-i+1} \frac{1}{j}$$

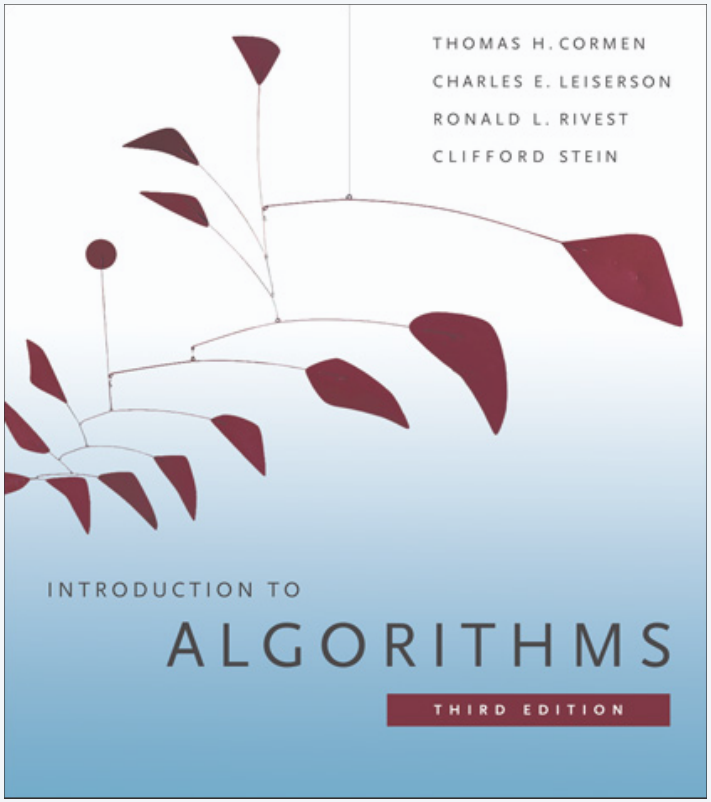
all pairs i and j

$$\leq 2N \sum_{j=1}^{N} \frac{1}{j}$$

$$\sim 2N \int_{x=1}^{N} \frac{1}{x} \, dx$$

$$= 2N \ln N$$

Remark. Number of compares only decreases if equal elements.



CHAPTER 9

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n + 1) / 2 \rfloor$.
- $O(n)$ compares for min or max.
- $O(n \log n)$ compares by sorting.
- $O(n \log k)$ compares with a binary heap.

Applications. Order statistics; find the "top k"; bottleneck paths, ...

Q. Can we do it with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

Quickselect

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Recur in one subarray—the one containing the k^{th} smallest element.

QUICK-SELECT (A, k)

Pick pivot $p \in A$ uniformly at random.

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, p).

3-way partitioning can be done in-place (using n–1 compares)

IF $k \leq |L|$ RETURN QUICK-SELECT (L, k).

ELSE IF $k > |L| + |M|$ RETURN QUICK-SELECT $(R, k - |L| - |M|)$

ELSE RETURN p.

Quickselect analysis

Intuition. Split candy bar uniformly $\Rightarrow$ expected size of larger piece is ¾.

$$T(n) \le T(\tfrac{3}{4}\, n) + n \quad \Rightarrow \quad T(n) \le 4\, n$$

Def. $T(n, k)$ = expected # compares to select k^{th} smallest in an array of size $\le n$.

Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \le 4\, n$.

Pf. [by strong induction on n]

- Assume true for $1, 2, \ldots, n-1$.
- $T(n)$ satisfies the following recurrence:

can assume we always recur on largest subarray since T(n) is monotonic and we are trying to get an upper bound

$$\begin{aligned} T(n) &\le n + 2 / n \left[T(n/2) + \ldots + T(n-3) + T(n-2) + T(n-1) \right] \\ &\le n + 2 / n \left[4\, n/2 + \ldots + 4(n-3) + 4(n-2) + 4(n-1) \right] \\ &= n + 4\,(3/4\, n) \\ &= 4\, n. \quad \blacksquare \end{aligned}$$

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst case linear time

Goal. Find pivot element p that divides list of n elements into two pieces so that each piece is guaranteed to have $\leq 7/10\, n$ elements.

Q. How to find approximate median in linear time?

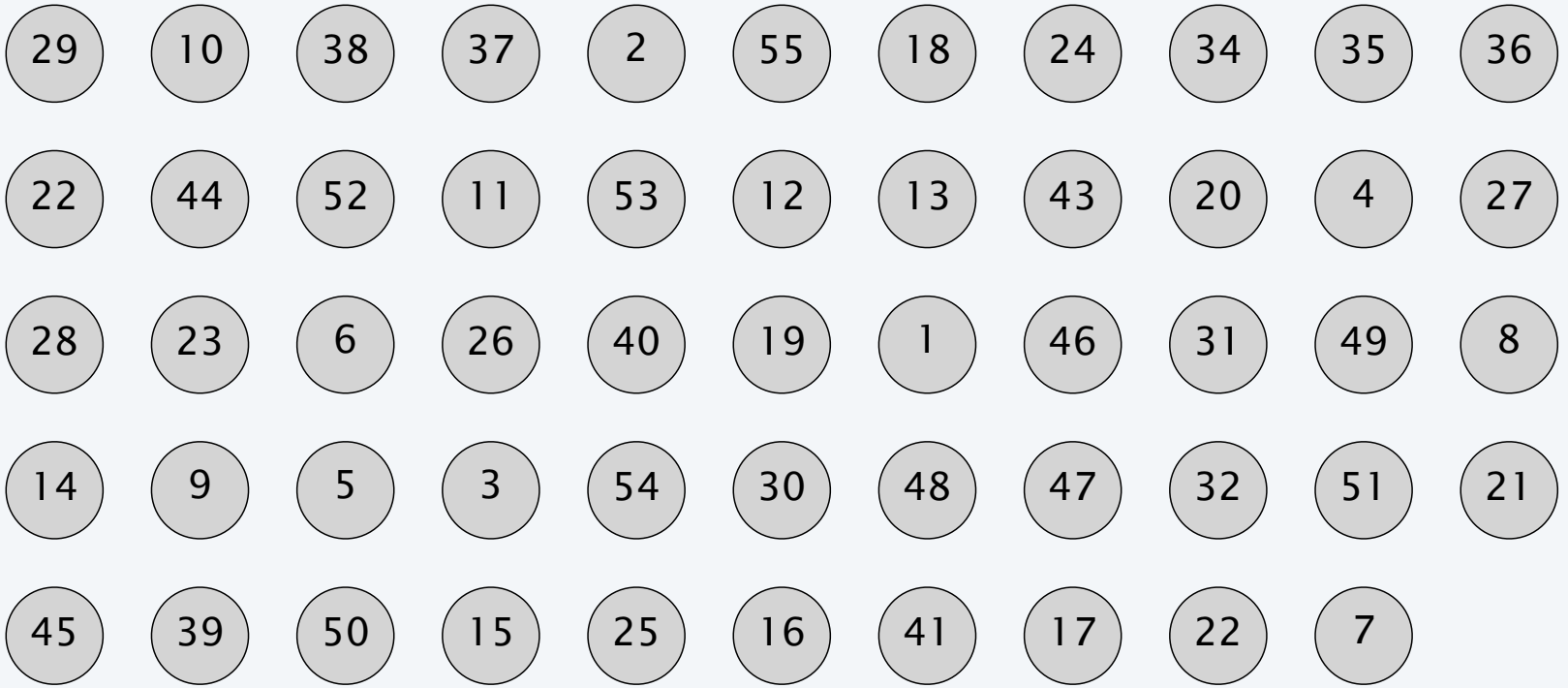
A. Recursively compute median of sample of $\leq 2/10\, n$ elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(7/10\, n) + T(2/10\, n) + \Theta(n) & \text{otherwise} \end{cases}$$

two subproblems of different sizes!

Choosing the pivot element

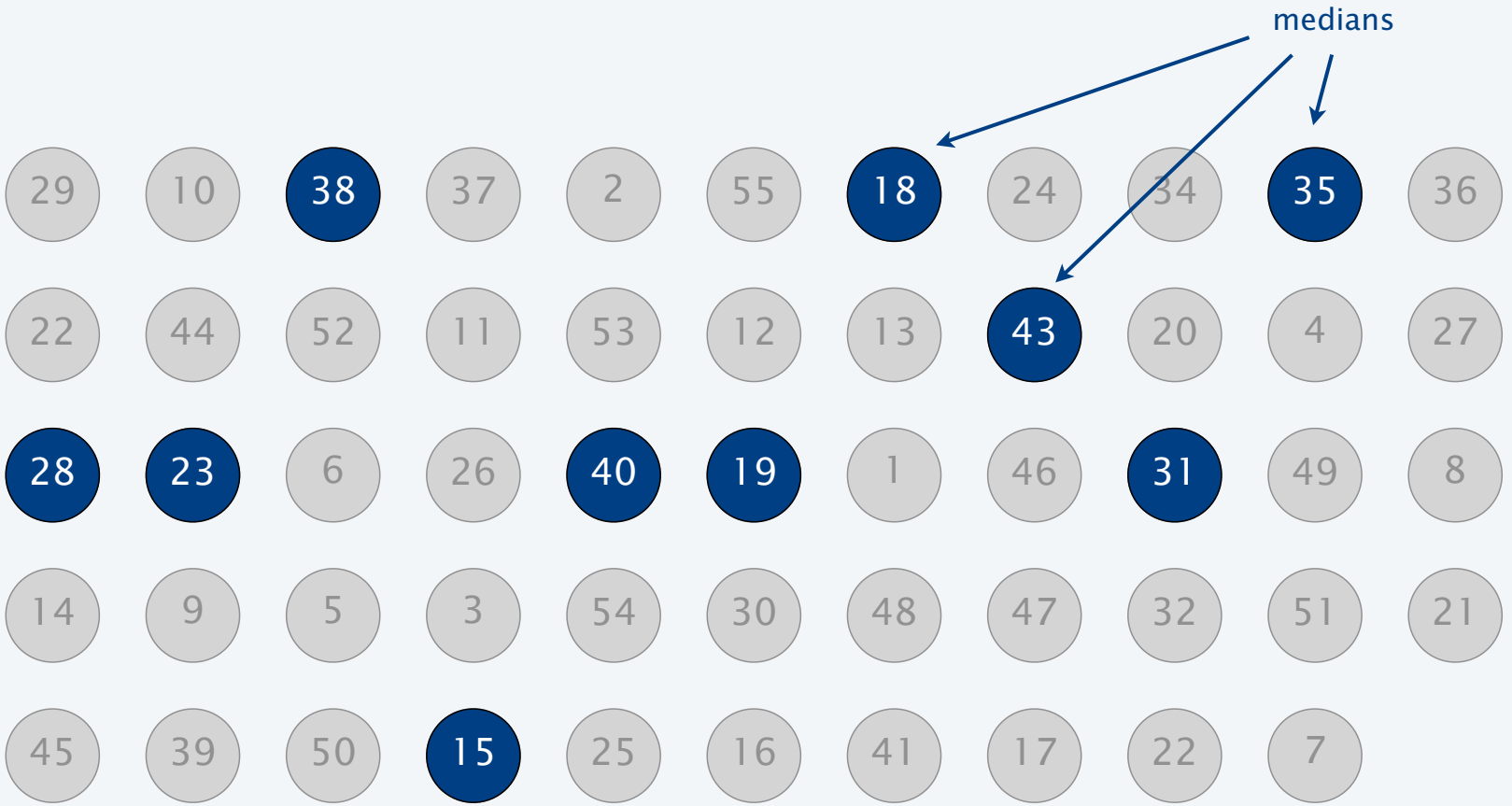
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).



N = 54

Choosing the pivot element

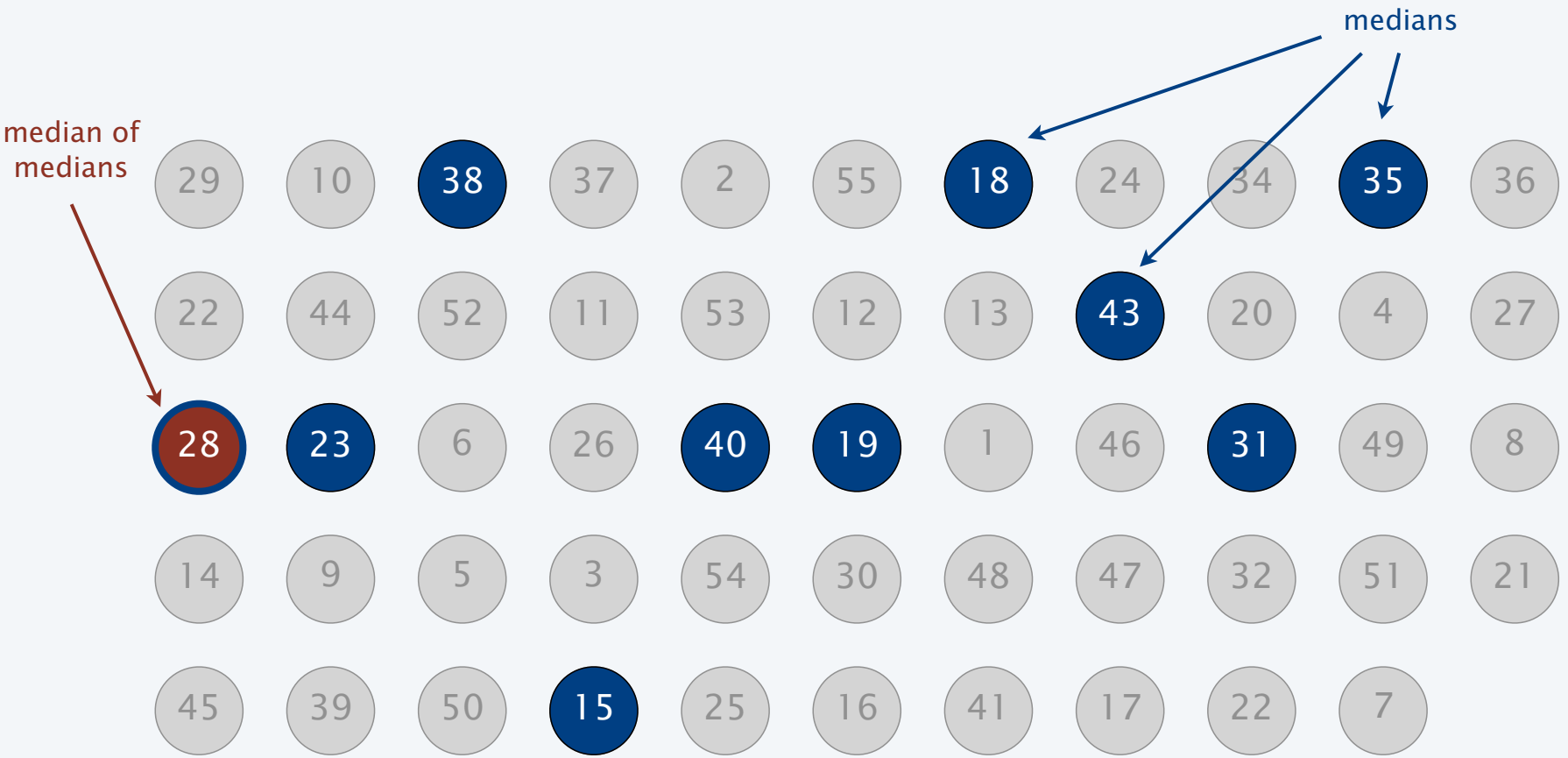
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).



N = 54

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n / 5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.



N = 54

Median-of-medians selection algorithm

MOM-SELECT (A, k)

$n \leftarrow |A|$.

IF $n < 50$ RETURN k^{th} smallest of element of A via mergesort.

Group A into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).

$B \leftarrow$ median of each group of 5.

$p \leftarrow$ MOM-SELECT$(B, \lfloor n / 10 \rfloor)$ ← median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, p).

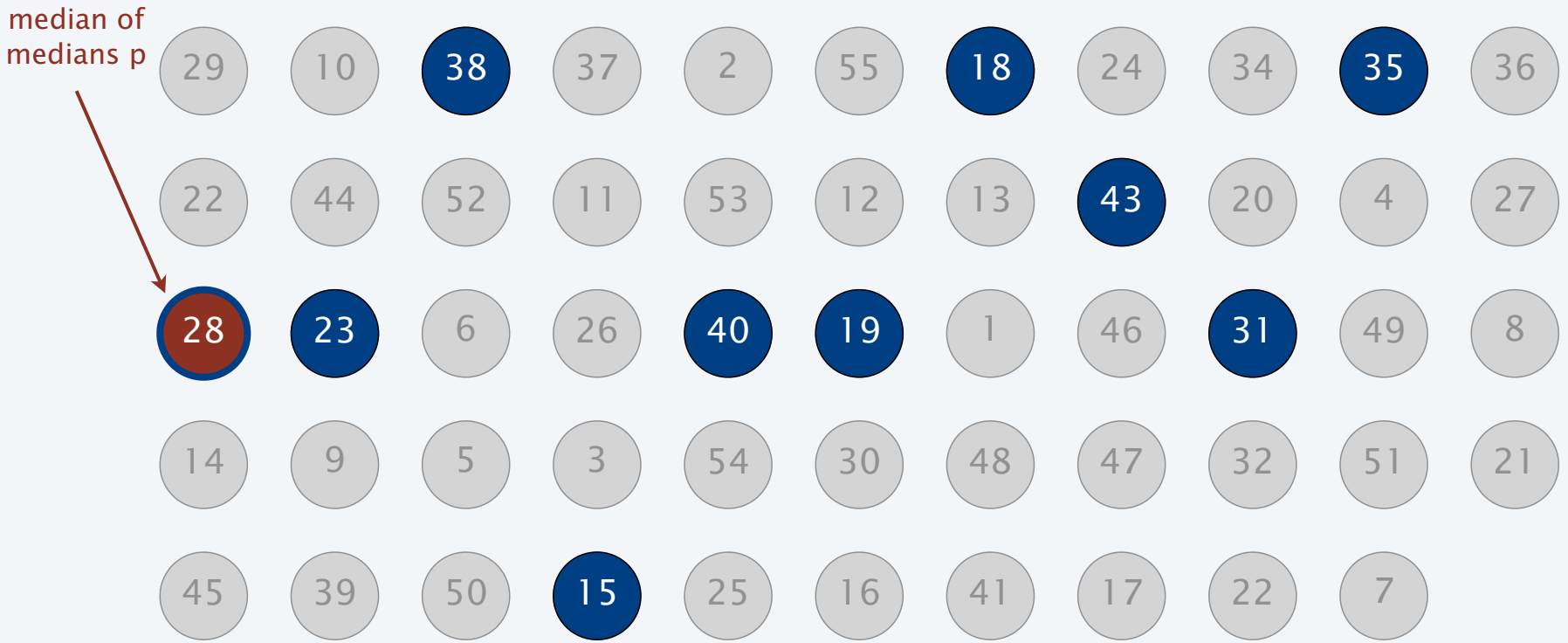
IF $k \leq |L|$ RETURN MOM-SELECT (L, k).

ELSE IF $k > |L| + |M|$ RETURN MOM-SELECT $(R, k - |L| - |M|)$

ELSE RETURN p.

Analysis of median-of-medians selection algorithm

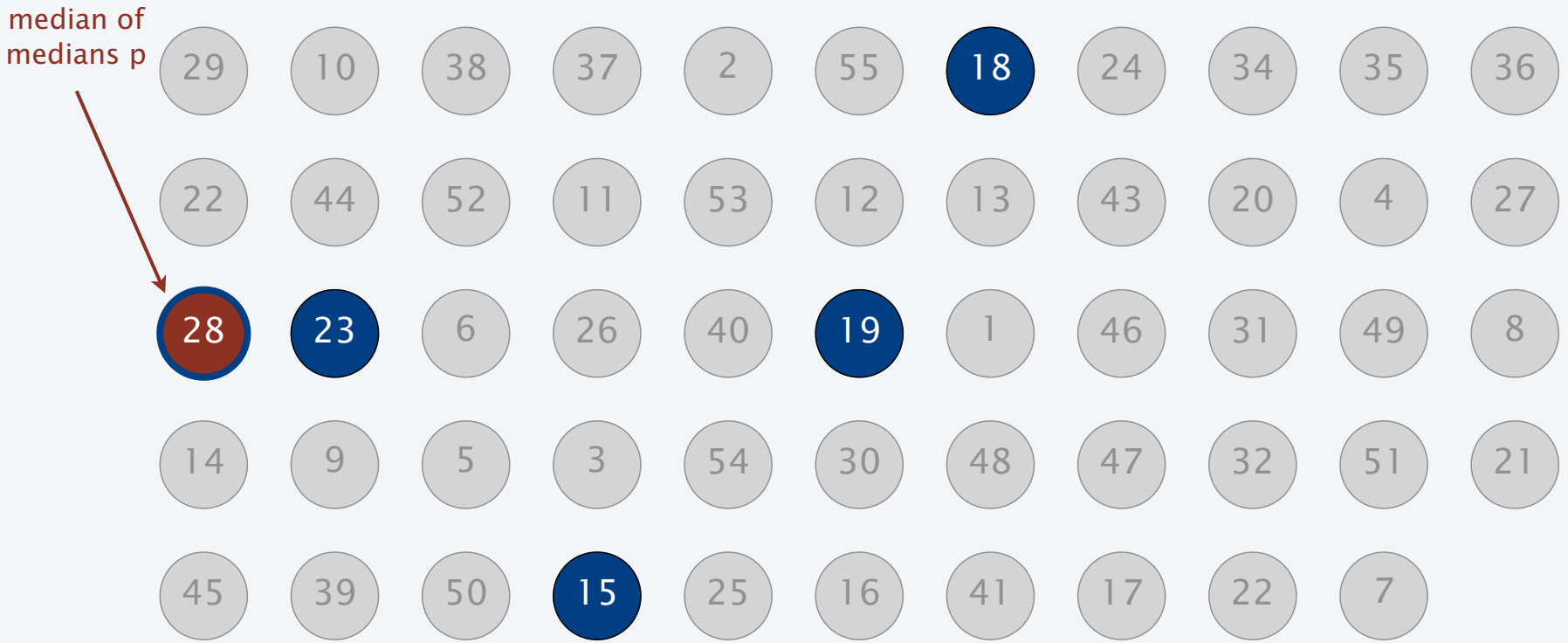
- At least half of 5-element medians $\leq p$.



N = 54

Analysis of median-of-medians selection algorithm

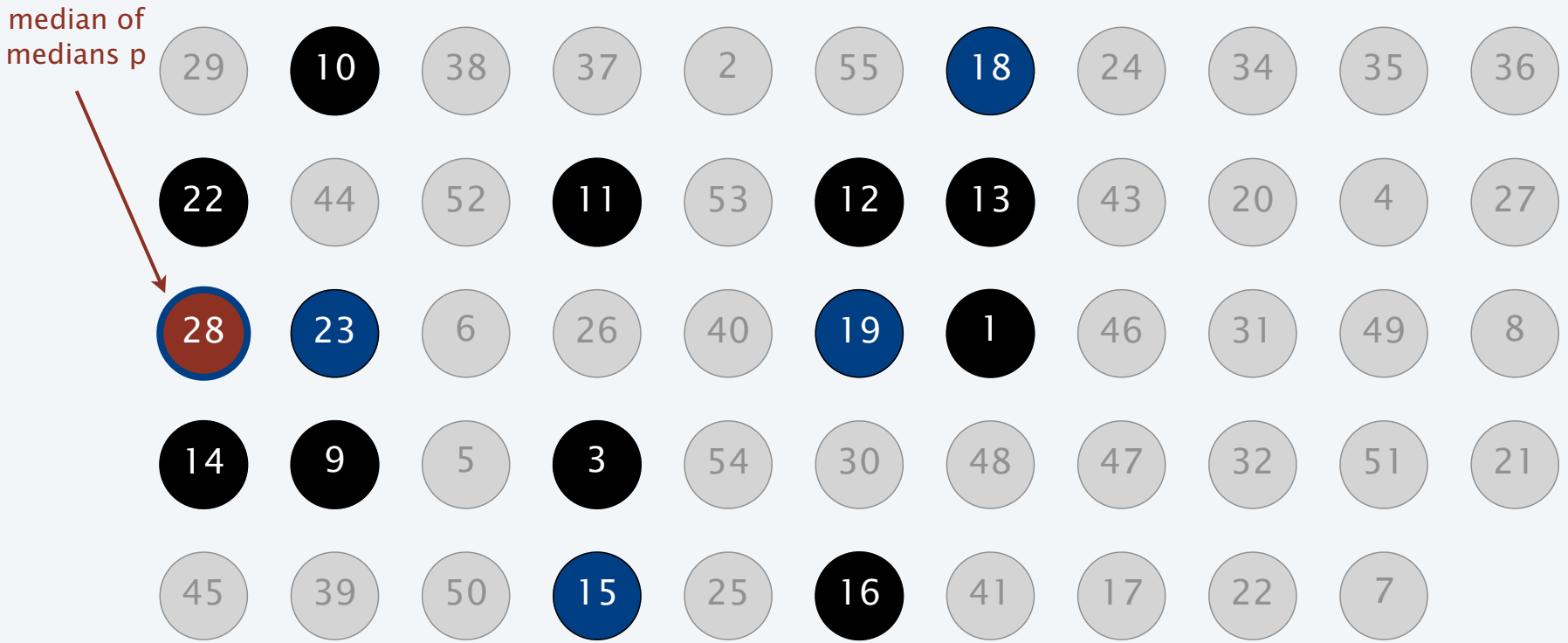
- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ medians $\leq p$.



N = 54

Analysis of median-of-medians selection algorithm

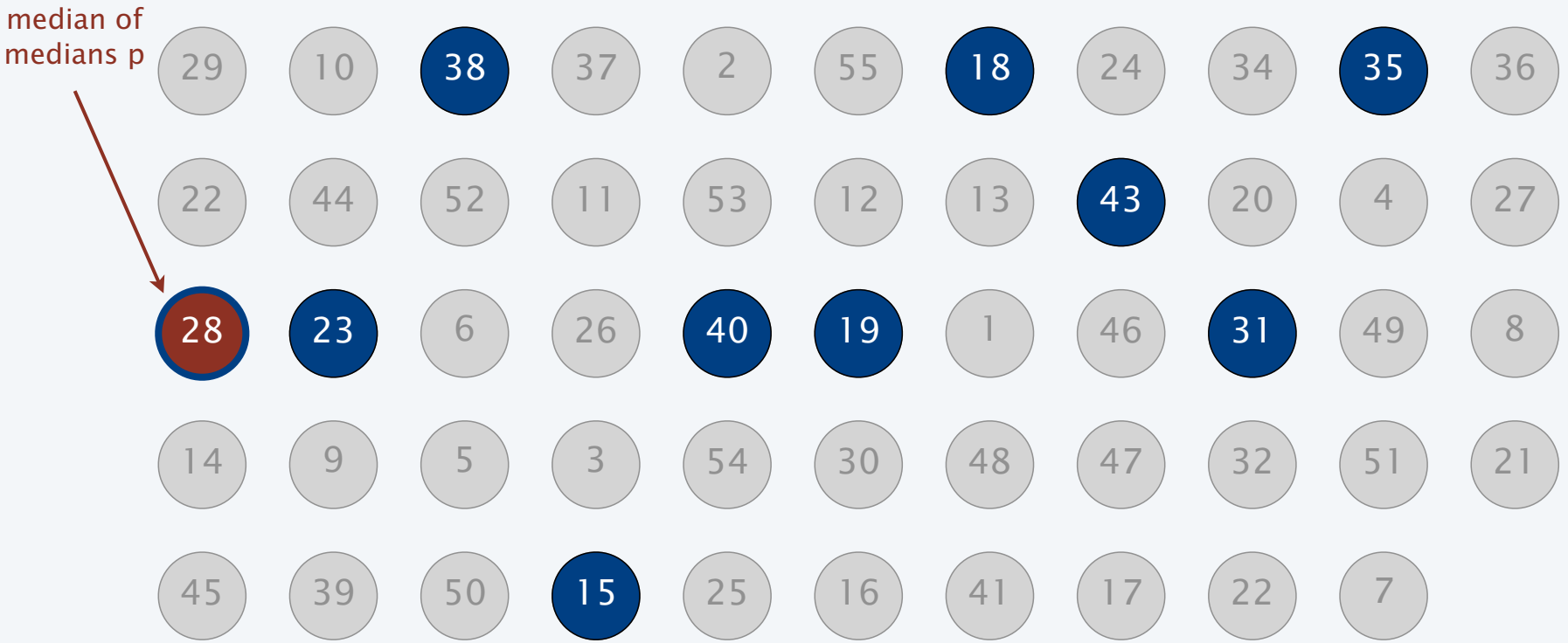
- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ medians $\leq p$.
- At least $3 \lfloor n/10 \rfloor$ elements $\leq p$.



N = 54

Analysis of median-of-medians selection algorithm

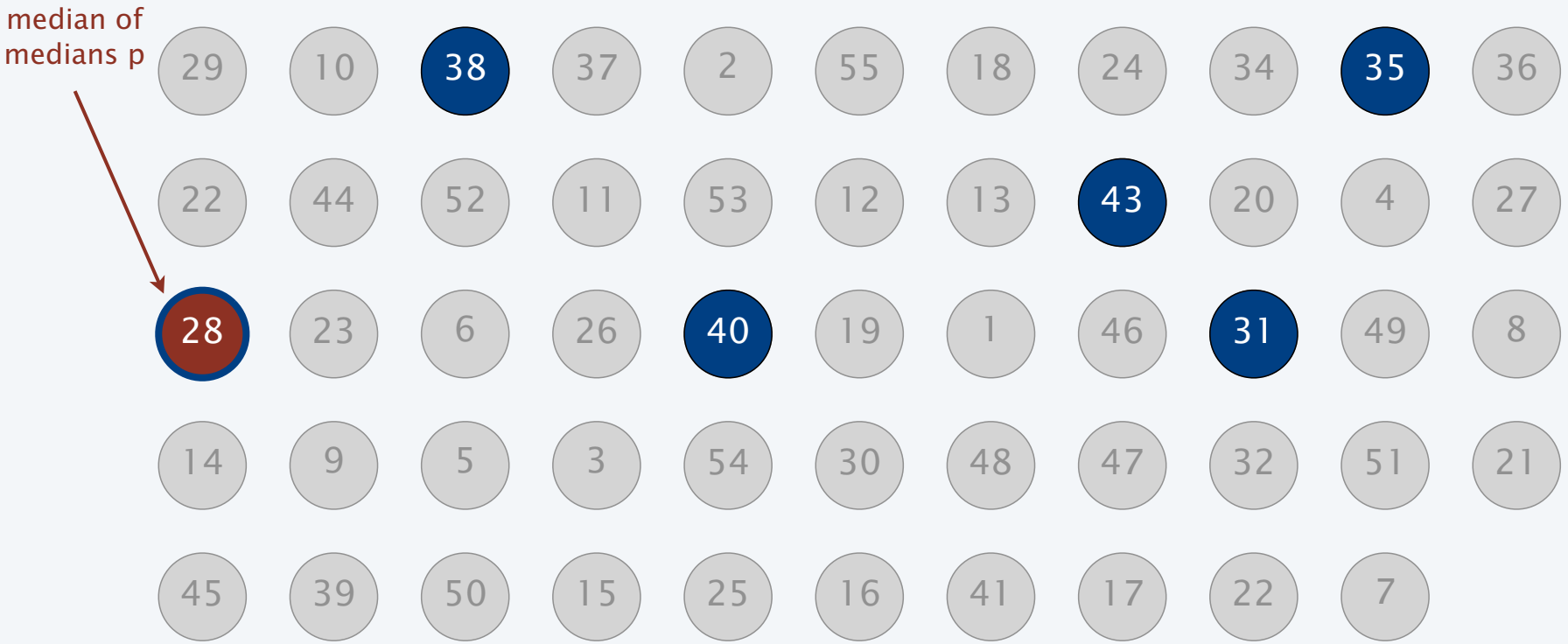
- At least half of 5-element medians ≥ p.



N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.
- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians ≥ p.



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Analysis of median-of-medians selection algorithm

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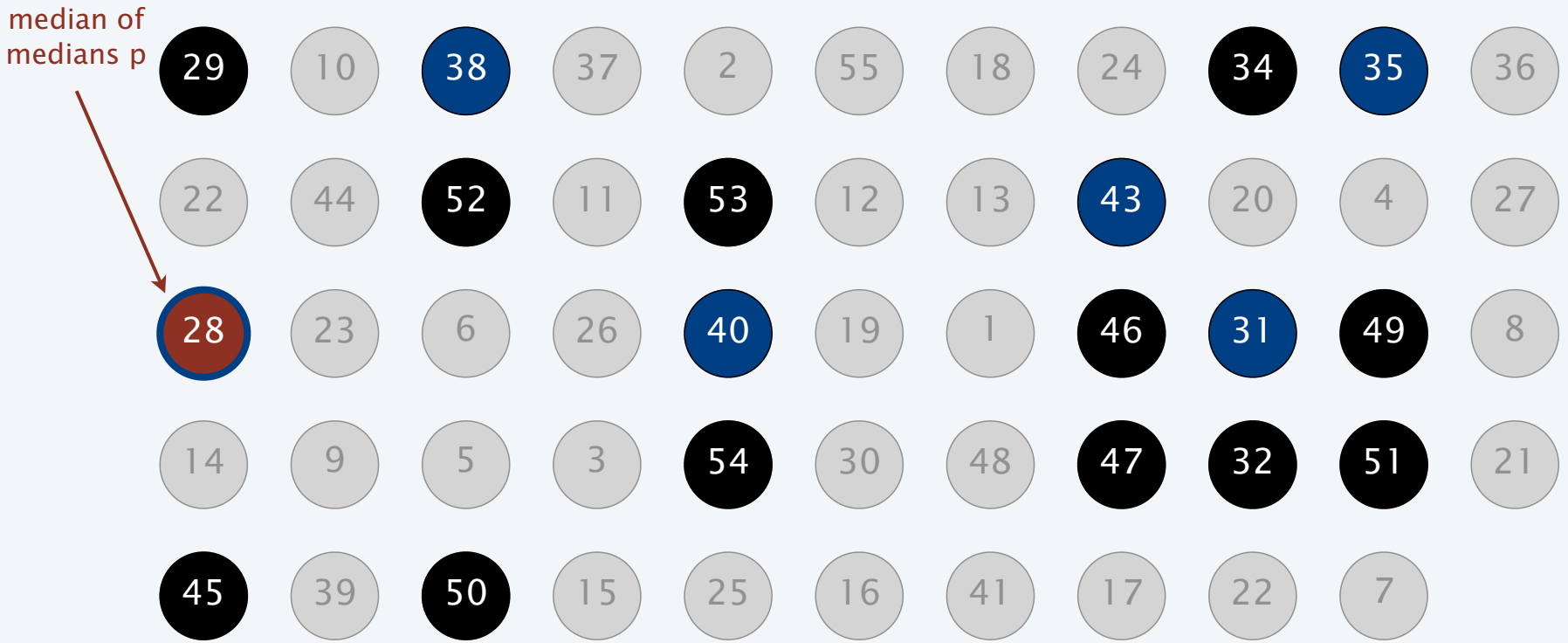


N = 54

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on an array of n elements.

$$C(n) \leq C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + {}^{11}\!/_{5}\, n$$

median of medians

recursive select

computing median of 5 (6 compares per group)

partitioning (n compares)

Now, solve recurrence.

- Assume n is both a power of 5 and a power of 10?
- Assume $C(n)$ is monotone nondecreasing?

Median-of-medians selection algorithm recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # compares on an array of $\le n$ elements.
- $T(n)$ is monotone, but $C(n)$ is not!

$$\mathrm{T}(n) \le \begin{cases} 6n & \text{if } n < 50 \\ T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5\, n & \text{otherwise} \end{cases}$$

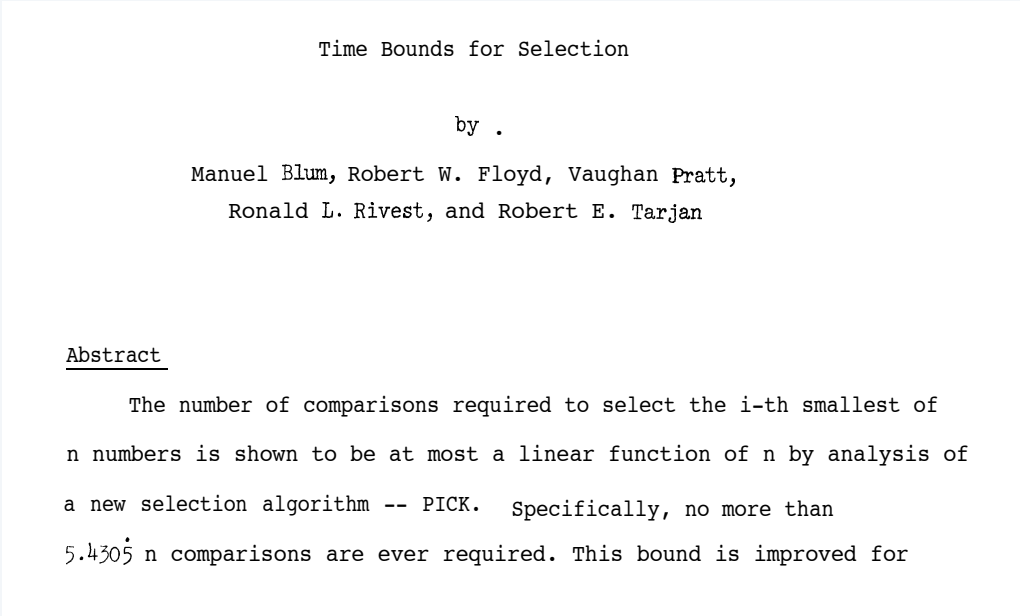
Claim. $T(n) \le 44\, n$.

- Base case: $T(n) \le 6n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- Induction step: for $n \ge 50$, we have:

$$\begin{aligned} T(n) &\le T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5\, n \\ &\le 44\,(\lfloor n/5 \rfloor) + 44\,(n - 3\lfloor n/10 \rfloor) + 11/5\, n \\ &\le 44\,(n/5) + 44\, n - 44\,(n/4) + 11/5\, n \quad \longleftarrow \text{ for } n \ge 50,\ 3\lfloor n/10 \rfloor \ge n/4 \\ &= 44\, n. \quad \blacksquare \end{aligned}$$

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.



Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Theory.

- Optimized version of BFPRT: $\leq 5.4305\, n$ compares.
- Best known upper bound [Dor-Zwick 1995]: $\leq 2.95\, n$ compares.
- Best known lower bound [Dor-Zwick 1999]: $\geq (2 + \varepsilon)\, n$ compares.

Linear-time selection postmortem

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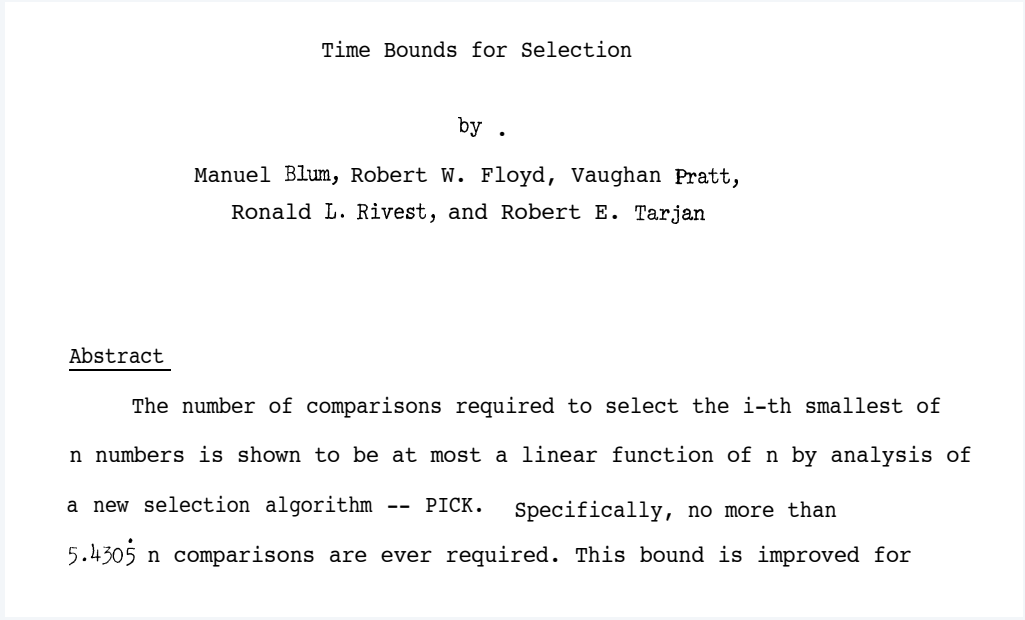

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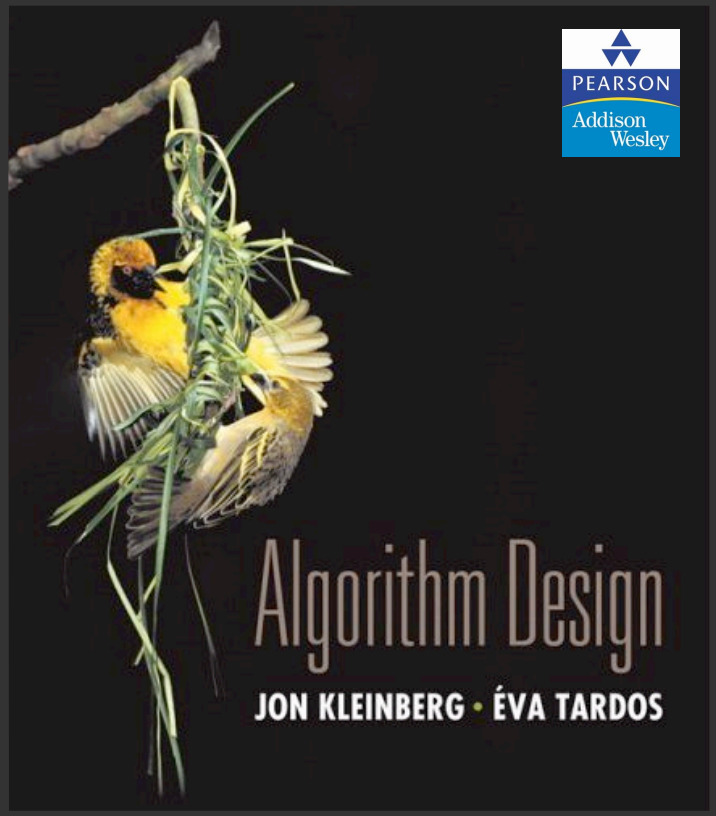
Practice. Constant and overhead (currently) too large to be useful.

Open. Practical selection algorithm whose worst-case running time is $O(n)$.



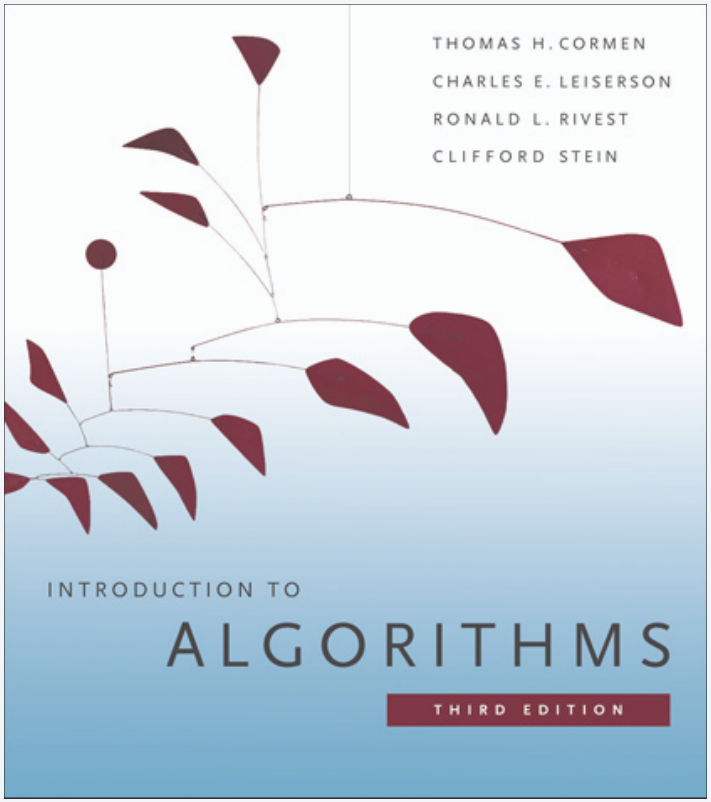
Divide and Conquer II

- *master theorem*
- *integer multiplication*
- *matrix multiplication*
- *convolution and FFT*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on Sep 8, 2013 6:33 AM



SECTIONS 4.3–4.6

DIVIDE AND CONQUER II

- *master theorem*
- *integer multiplication*
- *matrix multiplication*
- *convolution and FFT*

Master method

Goal. Recipe for solving common divide-and-conquer recurrences:

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

Terms.

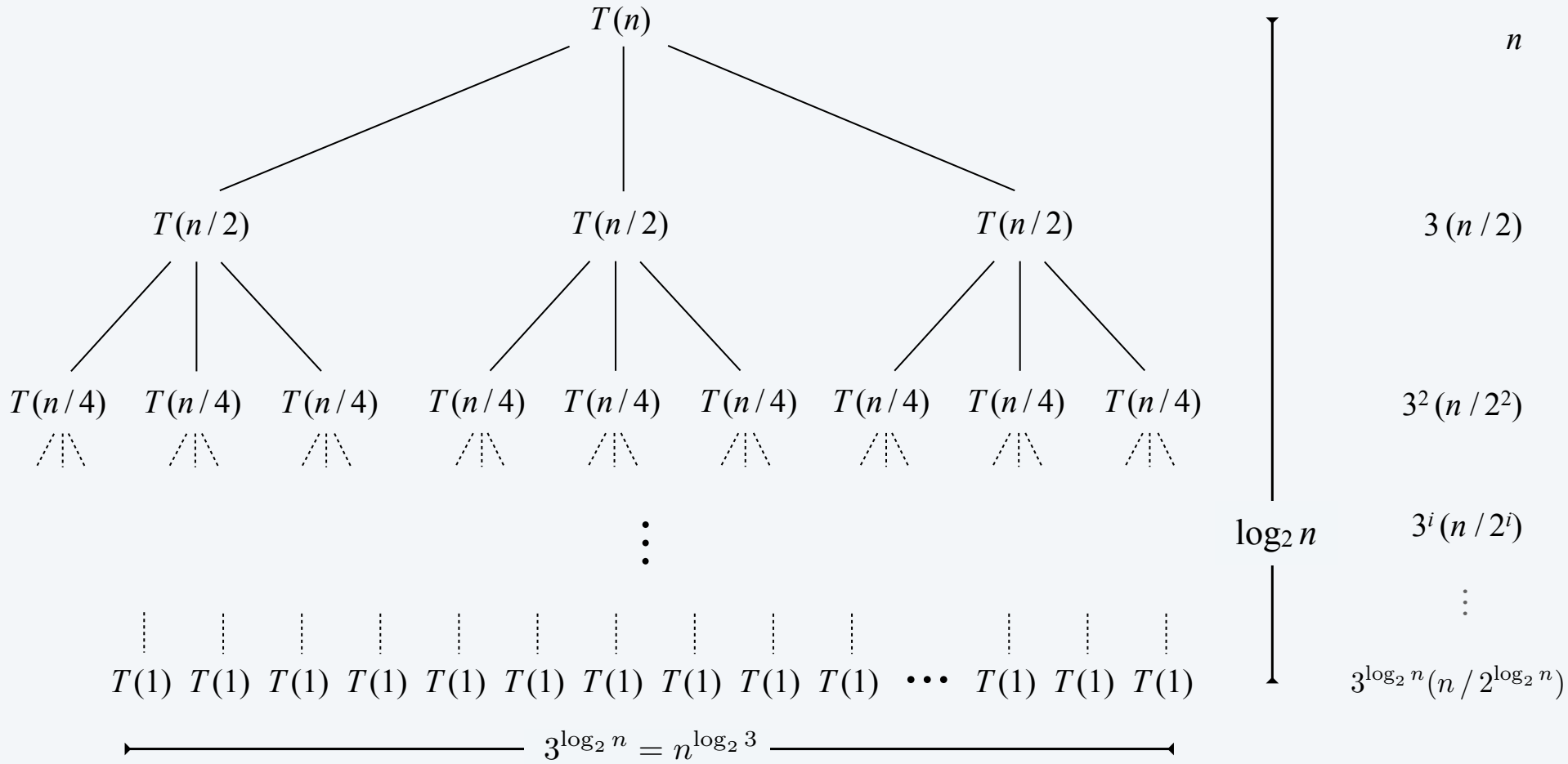
- $a \geq 1$ is the number of subproblems.
- $b > 0$ is the factor by which the subproblem size decreases.
- $f(n) =$ work to divide/merge subproblems.

Recursion tree.

- $k = \log_b n$ levels.
- $a^i =$ number of subproblems at level i.
- $n / b^i =$ size of subproblem at level i.

Case 1: total cost dominated by cost of leaves

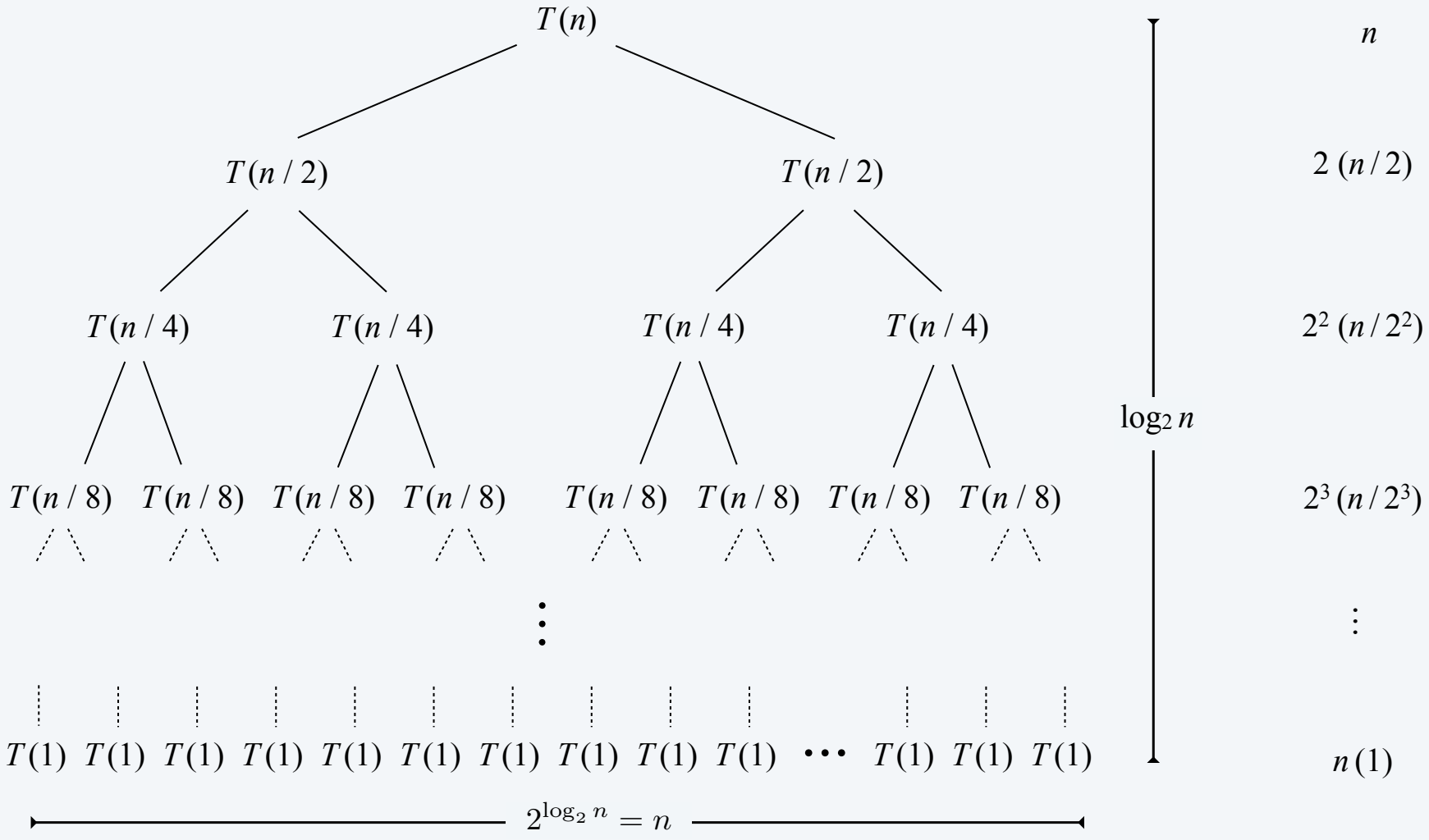
Ex 1. If $T(n)$ satisfies $T(n) = 3\,T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n^{\lg 3})$.



$$r = 3/2 > 1 \qquad T(n) = (1 + r + r^2 + r^3 + \ldots + r^{\log_2 n})\, n = \frac{r^{1+\log_2 n} - 1}{r - 1}\, n = 3n^{\log_2 3} - 2n$$

Case 2: total cost evenly distributed among levels

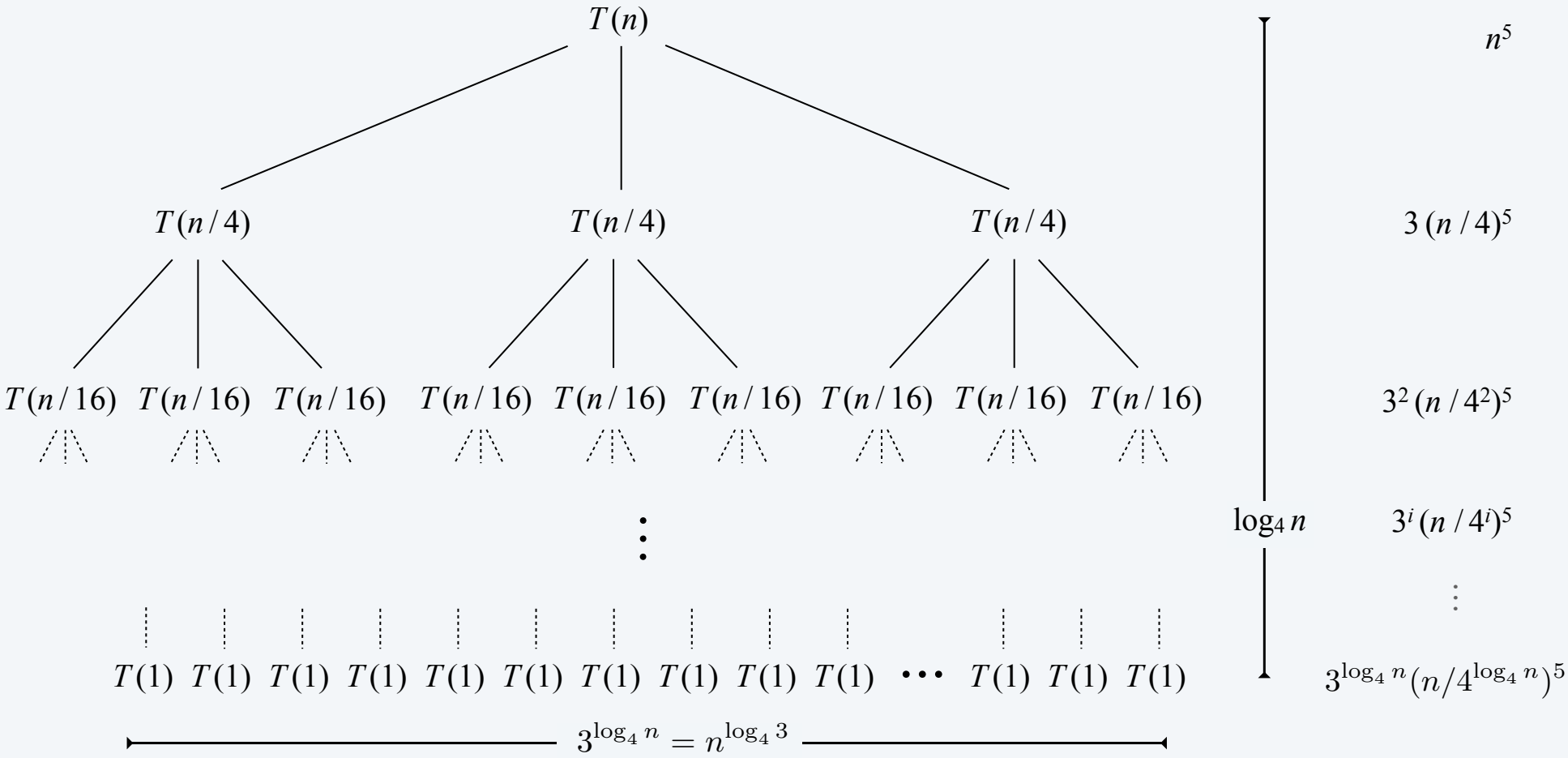
Ex 2. If $T(n)$ satisfies $T(n) = 2\,T(n/2) + n$, with $T(1) = 1$, then $T(n) = \Theta(n \log n)$.



$r = 1$ $\qquad T(n) = (1 + r + r^2 + r^3 + \ldots + r^{\log_2 n})\, n \;=\; n\,(\log_2 n + 1)$

Case 3: total cost dominated by cost of root

Ex 3. If $T(n)$ satisfies $T(n) = 3\,T(n/4) + n^5$, with $T(1) = 1$, then $T(n) = \Theta(n^5)$.



$$r = 3/4^5 < 1 \qquad n^5 \;\le\; T(n) \;\le\; (1 + r + r^2 + r^3 + \ldots)\, n^5 \;\le\; \frac{1}{1-r}\, n^5$$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Ex. $T(n) = 3\,T(n/2) + n$.

- $a = 3,\ b = 2,\ f(n) = n,\ \ k = \log_2 3$.
- $T(n) = \Theta(n^{\lg 3})$.

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Ex. $T(n) = 2\,T(n/2) + \Theta(n \log n)$.

- $a = 2,\ b = 2,\ f(n) = 17n,\ \ k = \log_2 2 = 1,\ p = 1.$
- $T(n) = \Theta(n \log^2 n)$.

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

regularity condition holds if $f(n) = \Theta(n^{k+\varepsilon})$

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Ex. $T(n) = 3\,T(n/4) + n^5$.

- $a = 3,\ b = 4,\ f(n) = n^5,\ k = \log_4 3$.
- $T(n) = \Theta(n^5)$.

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers that satisfies the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Let $k = \log_b a$. Then,

Case 1. If $f(n) = O(n^{k-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^k)$.

Case 2. If $f(n) = \Theta(n^k \log^p n)$, then $T(n) = \Theta(n^k \log^{p+1} n)$.

Case 3. If $f(n) = \Omega(n^{k+\varepsilon})$ for some constant $\varepsilon > 0$ and if $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Pf sketch.

- Use recursion tree to sum up terms (assuming n is an exact power of b).
- Three cases for geometric series.
- Deal with floors and ceilings.

Akra-Bazzi theorem

Desiderata. Generalizes master theorem to divide-and-conquer algorithms where subproblems have substantially different sizes.

Theorem. [Akra-Bazzi] Given constants $a_i > 0$ and $0 < b_i \le 1$, functions $h_i(n) = O(n / \log^2 n)$ and $g(n) = O(n^c)$, if the function $T(n)$ satisfies the recurrence:

$$T(n) = \sum_{i=1}^{k} a_i \, T(b_i n + h_i(n)) + g(n)$$

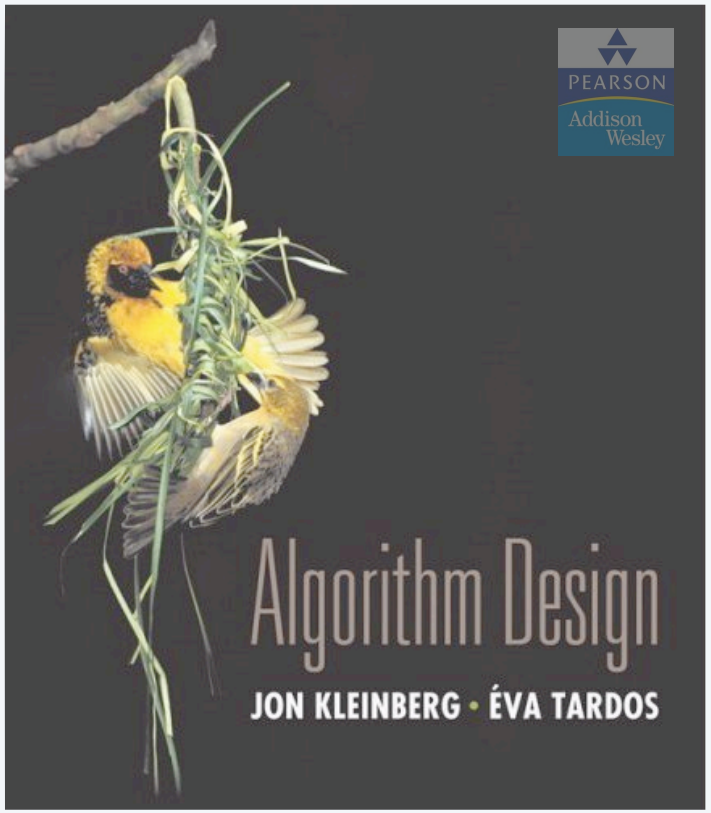
a_i subproblems of size b_i n

small perturbation to handle floors and ceilings

Then $T(n) = \Theta\left(n^p \left(1 + \int_1^n \frac{g(u)}{u^{p+1}} du\right)\right)$ where p satisfies $\sum_{i=1}^{k} a_i \, b_i^p = 1$.

Ex. $T(n) = 7/4 \; T(\lfloor n/2 \rfloor) + T(\lceil 3/4 \, n \rceil) + n^2$.

- $a_1 = 7/4, \; b_1 = 1/2, \; a_2 = 1, \; b_2 = 3/4 \;\Rightarrow\; p = 2.$
- $h_1(n) = \lfloor 1/2 \, n \rfloor - 1/2 \, n, \;\; h_2(n) = \lceil 3/4 \, n \rceil - 3/4 \, n.$
- $g(n) = n^2 \;\Rightarrow\; T(n) = \Theta(n^2 \log n).$



SECTION 5.5

DIVIDE AND CONQUER II

- *master theorem*
- *integer multiplication*
- *matrix multiplication*
- *convolution and FFT*

Integer addition

Addition. Given two n-bit integers a and b, compute $a + b$.

Subtraction. Given two n-bit integers a and b, compute $a - b$.

Grade-school algorithm. $\Theta(n)$ bit operations.

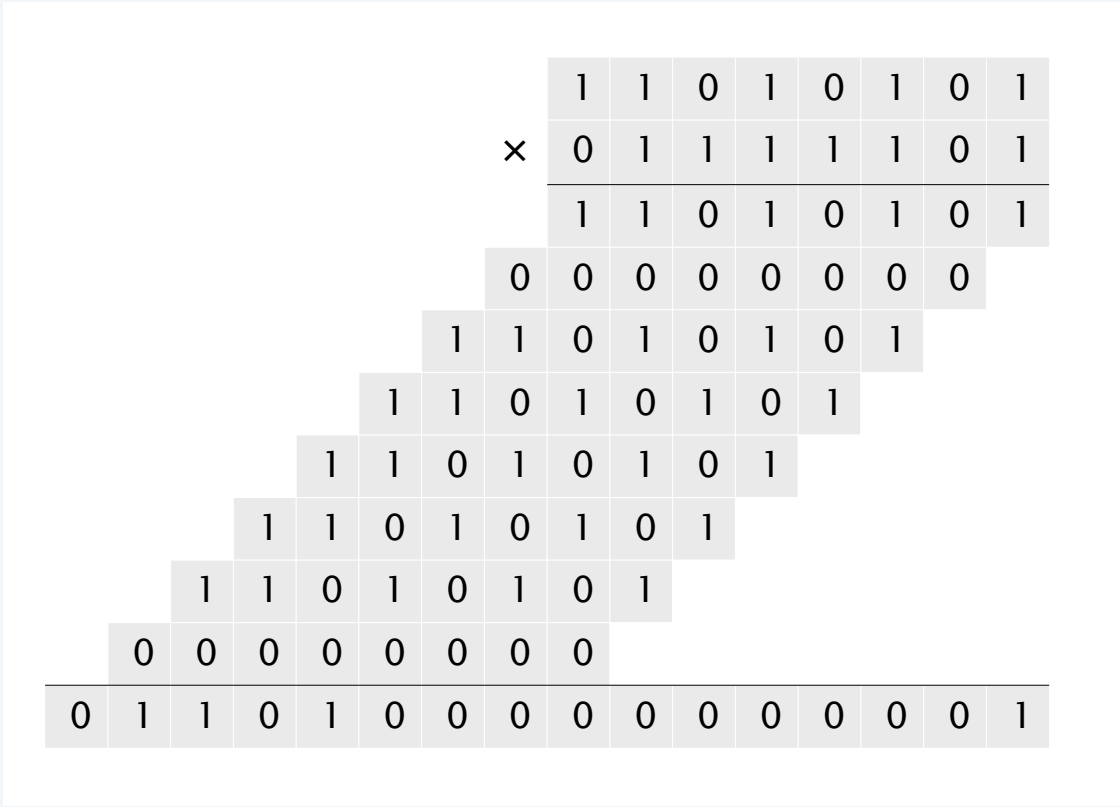
	1	1	1	1	1	1	0	1
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Remark. Grade-school addition and subtraction algorithms are asymptotically optimal.

Integer multiplication

Multiplication. Given two n-bit integers a and b, compute $a \times b$.

Grade-school algorithm. $\Theta(n^2)$ bit operations.



Conjecture. [Kolmogorov 1952] Grade-school algorithm is optimal.

Theorem. [Karatsuba 1960] Conjecture is wrong.

Divide-and-conquer multiplication

To multiply two n-bit integers x and y:

- Divide x and y into low- and high-order bits.
- Multiply four $\frac{1}{2}n$-bit integers, recursively.
- Add and shift to obtain result.

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

← use bit shifting to compute 4 terms

$$(2^m a + b)(2^m c + d) = 2^{2m} ac + 2^m (bc + ad) + bd$$

1 2 3 4

Ex. $x = 1\,0\,0\,0\,1\,1\,0\,1$ $y = 1\,1\,1\,0\,0\,0\,0\,1$

a b c d

Divide-and-conquer multiplication

MULTIPLY(x, y, n)

IF $(n = 1)$

  RETURN $x \times y$.

ELSE

  $m \leftarrow \lceil n / 2 \rceil$.

  $a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$.

  $c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

  $e \leftarrow$ MULTIPLY(a, c, m).

  $f \leftarrow$ MULTIPLY(b, d, m).

  $g \leftarrow$ MULTIPLY(b, c, m).

  $h \leftarrow$ MULTIPLY(a, d, m).

  RETURN $2^{2m} e + 2^m (g + h) + f$.

Divide-and-conquer multiplication analysis

Proposition. The divide-and-conquer multiplication algorithm requires $\Theta(n^2)$ bit operations to multiply two n-bit integers.

Pf. Apply case 1 of the master theorem to the recurrence:

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

Karatsuba trick

To compute middle term $bc + ad$, use identity:

$$bc + ad = ac + bd - (a - b)(c - d)$$

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

middle term

$$(2^m a + b)(2^m c + d) = 2^{2m} ac + 2^m (bc + ad) + bd$$

$$= 2^{2m} ac + 2^m (ac + bd - (a - b)(c - d)) + bd$$

1 1 3 2 3

Bottom line. Only three multiplication of $n/2$-bit integers.

Karatsuba multiplication

KARATSUBA-MULTIPLY(x, y, n)

IF $(n = 1)$

  RETURN $x \times y$.

ELSE

  $m \leftarrow \lceil\, n / 2 \,\rceil$.

  $a \leftarrow \lfloor\, x / 2^m \,\rfloor$; $b \leftarrow x \bmod 2^m$.

  $c \leftarrow \lfloor\, y / 2^m \,\rfloor$; $d \leftarrow y \bmod 2^m$.

  $e \leftarrow$ KARATSUBA-MULTIPLY(a, c, m).

  $f \leftarrow$ KARATSUBA-MULTIPLY(b, d, m).

  $g \leftarrow$ KARATSUBA-MULTIPLY$(a - b, c - d, m)$.

  RETURN $2^{2m}\, e + 2^m\, (e + f - g) + f$.

Karatsuba analysis

Proposition. Karatsuba's algorithm requires $O(n^{1.585})$ bit operations to multiply two n-bit integers.

Pf. Apply case 1 of the master theorem to the recurrence:

$$T(n) = 3\,T(n/2) + \Theta(n) \quad \Rightarrow \quad T(n) = \Theta(n^{\lg 3}) = O(n^{1.585}).$$

Practice. Faster than grade-school algorithm for about 320-640 bits.

Integer arithmetic reductions

Integer multiplication. Given two n-bit integers, compute their product.

problem	arithmetic	running time
integer multiplication	$a \times b$	$\Theta(M(n))$
integer division	$a / b,\ a \bmod b$	$\Theta(M(n))$
integer square	a^2	$\Theta(M(n))$
integer square root	$\lfloor \sqrt{a} \rfloor$	$\Theta(M(n))$

integer arithmetic problems with the same complexity as integer multiplication

History of asymptotic complexity of integer multiplication

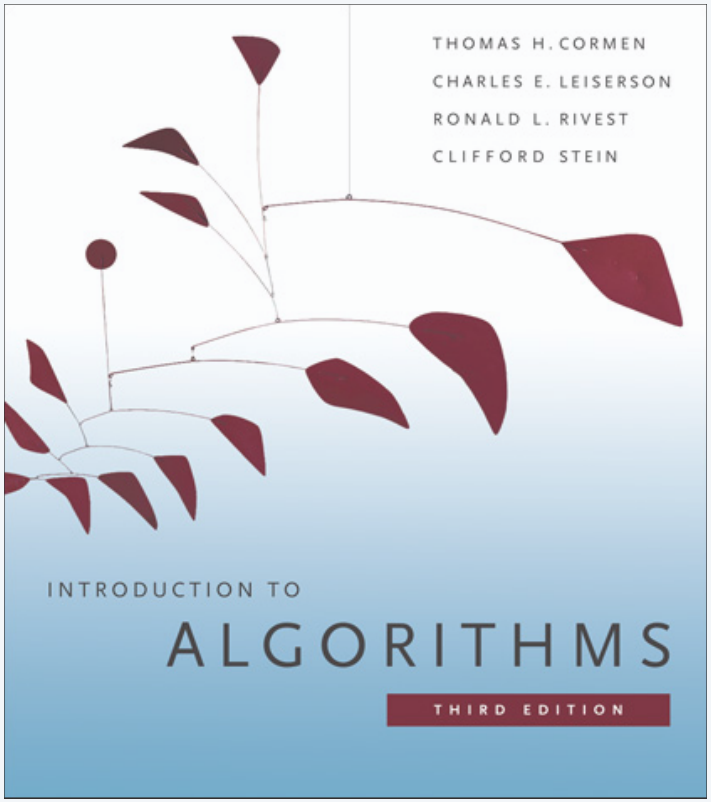
year	algorithm	order of growth
?	brute force	$\Theta(n^2)$
1962	Karatsuba-Ofman	$\Theta(n^{1.585})$
1963	Toom-3, Toom-4	$\Theta(n^{1.465})$, $\Theta(n^{1.404})$
1966	Toom-Cook	$\Theta(n^{1+\varepsilon})$
1971	Schönhage–Strassen	$\Theta(n \log n \log \log n)$
2007	Fürer	$n \log n \, 2^{O(\log^* n)}$
?	?	$\Theta(n)$

number of bit operations to multiply two n-bit integers

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»



SECTION 4.2

DIVIDE AND CONQUER II

- *master theorem*
- *integer multiplication*
- *matrix multiplication*
- *convolution and FFT*

Dot product

Dot product. Given two length n vectors a and b, compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$

$$a = \begin{bmatrix} .70 & .20 & .10 \end{bmatrix}$$

$$b = \begin{bmatrix} .30 & .40 & .30 \end{bmatrix}$$

$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Remark. Grade-school dot product algorithm is asymptotically optimal.

Matrix multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

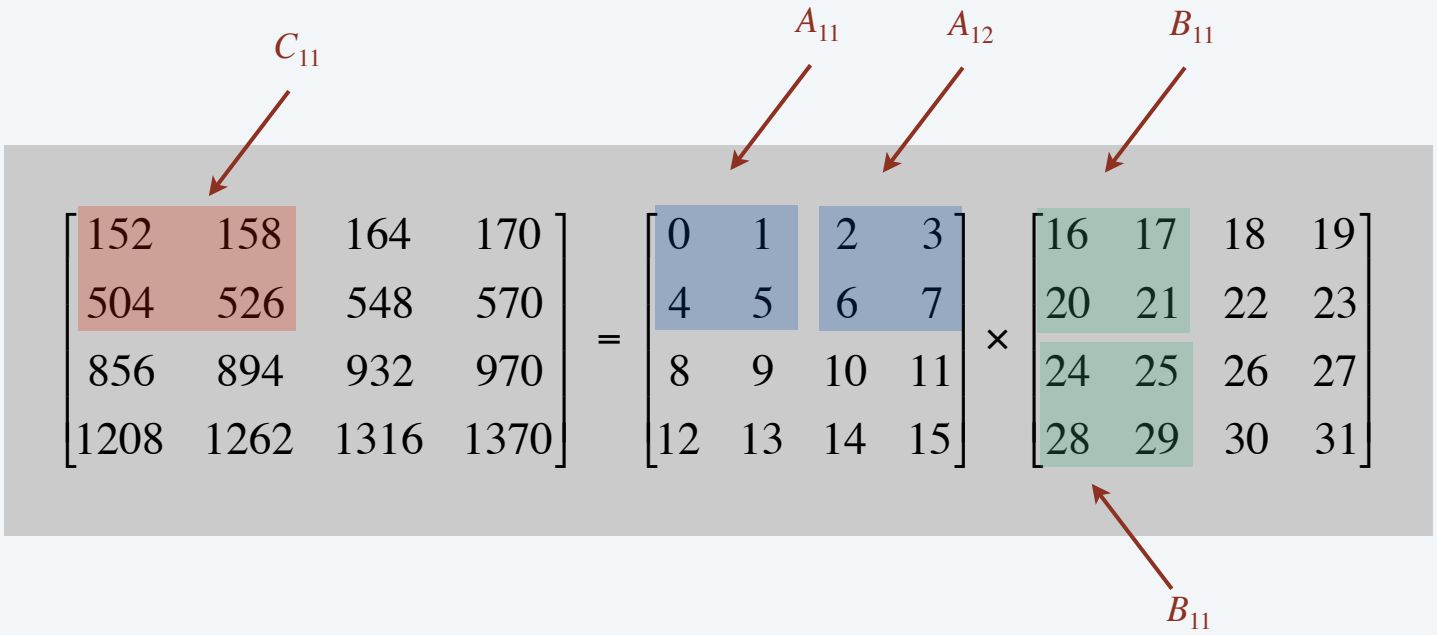
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm asymptotically optimal?

Block matrix multiplication



$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix multiplication: warmup

To multiply two n-by-n matrices A and B:

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

Running time. Apply case 1 of Master Theorem.

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Strassen's trick

Key idea. multiply 2-by-2 blocks with only 7 multiplications.
(plus 11 additions and 7 subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_1 + P_5 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &\leftarrow A_{11} \times (B_{12} - B_{22}) \\ P_2 &\leftarrow (A_{11} + A_{12}) \times B_{22} \\ P_3 &\leftarrow (A_{21} + A_{22}) \times B_{11} \\ P_4 &\leftarrow A_{22} \times (B_{21} - B_{11}) \\ P_5 &\leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &\leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &\leftarrow (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Pf. $C_{12} = P_1 + P_2$

$= A_{11} \times (B_{12} - B_{22}) + (A_{11} + A_{12}) \times B_{22}$

$= A_{11} \times B_{12} + A_{12} \times B_{22}$. ✔

Strassen's algorithm

assume n is a power of 2

STRASSEN (n, A, B)

IF $(n = 1)$ RETURN $A \times B$.

Partition A and B into 2-by-2 block matrices.

$P_1 \leftarrow$ STRASSEN $(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN $(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN $(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN $(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN $(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

keep track of indices of submatrices (don't copy matrix entries)

Analysis of Strassen's algorithm

Theorem. Strassen's algorithm requires $O(n^{2.81})$ arithmetic operations to multiply two n-by-n matrices.

Pf. Apply case 1 of the master theorem to the recurrence:

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Q. What if n is not a power of 2?

A. Could pad matrices with zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10 & 11 & 12 & 0 \\ 13 & 14 & 15 & 0 \\ 16 & 17 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 84 & 90 & 96 & 0 \\ 201 & 216 & 231 & 0 \\ 318 & 342 & 366 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Strassen's algorithm: practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm when n is "small" .

Common misperception. *"Strassen is only a theoretical curiosity."*

- Apple reports 8x speedup on G4 Velocity Engine when $n \approx 2{,}048$.
- Range of instances where it's useful is a subject of controversy.

Linear algebra reductions

Matrix multiplication. Given two n-by-n matrices, compute their product.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	$\Theta(MM(n))$
matrix inversion	A^{-1}	$\Theta(MM(n))$
determinant	$\lvert A \rvert$	$\Theta(MM(n))$
system of linear equations	$Ax = b$	$\Theta(MM(n))$
LU decomposition	$A = L\,U$	$\Theta(MM(n))$
least squares	$\min \lVert Ax - b \rVert_2$	$\Theta(MM(n))$

numerical linear algebra problems with the same complexity as matrix multiplication

Fast matrix multiplication: theory

Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

A. Yes! [Strassen 1969] $\Theta(n^{\log_2 7}) = O(n^{2.807})$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Multiply two 3-by-3 matrices with 21 scalar multiplications?

A. Unknown. $\Theta(n^{\log_3 21}) = O(n^{2.77})$

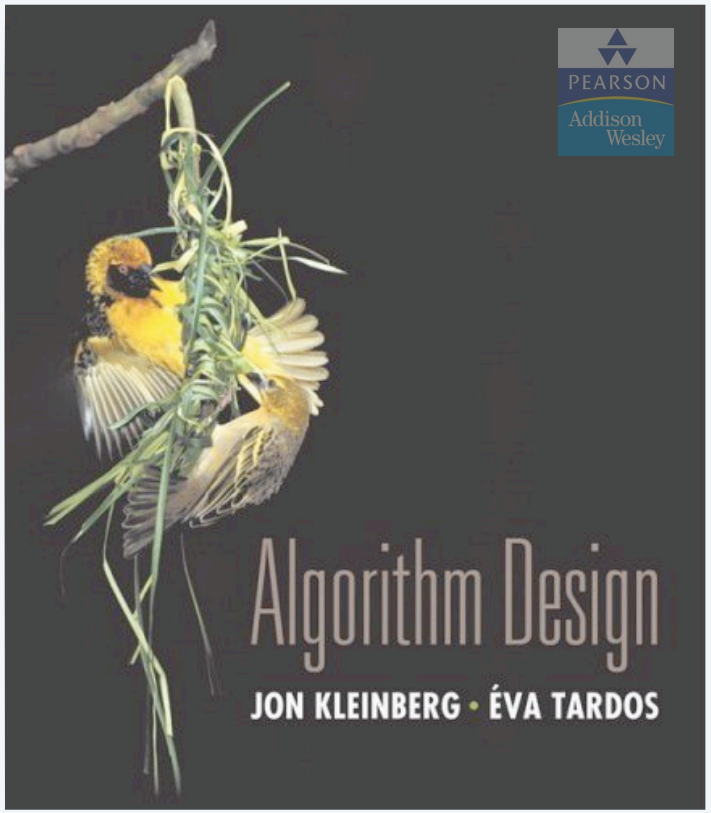
Begun, the decimal wars have. [Pan, Bini et al, Schönhage, ...]

- Two 20-by-20 matrices with 4,460 scalar multiplications. $O(n^{2.805})$
- Two 48-by-48 matrices with 47,217 scalar multiplications. $O(n^{2.7801})$
- A year later. $O(n^{2.7799})$
- December 1979. $O(n^{2.521813})$
- January 1980. $O(n^{2.521801})$

History of asymptotic complexity of matrix multiplication

year	algorithm	order of growth
?	brute force	$O(n^3)$
1969	Strassen	$O(n^{2.808})$
1978	Pan	$O(n^{2.796})$
1979	Bini	$O(n^{2.780})$
1981	Schönhage	$O(n^{2.522})$
1982	Romani	$O(n^{2.517})$
1982	Coppersmith-Winograd	$O(n^{2.496})$
1986	Strassen	$O(n^{2.479})$
1989	Coppersmith-Winograd	$O(n^{2.376})$
2010	Strother	$O(n^{2.3737})$
2011	Williams	$O(n^{2.3727})$
?	?	$O(n^{2+\varepsilon})$

number of floating-point operations to multiply two n-by-n matrices



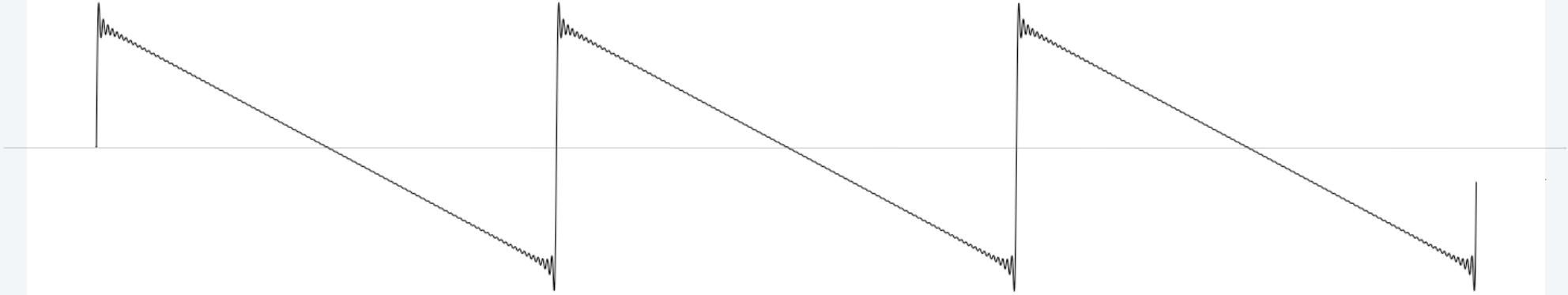
SECTION 5.6

DIVIDE AND CONQUER II

- *master theorem*
- *integer multiplication*
- *matrix multiplication*
- *convolution and FFT*

Fourier analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any (sufficiently smooth) periodic function can be expressed as the sum of a series of sinusoids.



$$y(t) = \frac{2}{\pi} \sum_{k=1}^{N} \frac{\sin kt}{k} \qquad N = 100$$

Euler's identity

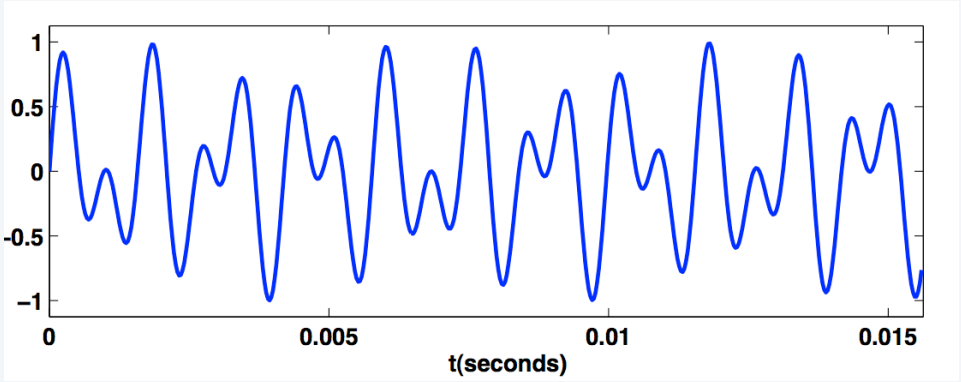
Euler's identity. $e^{ix} = \cos x + i \sin x$.

Sinusoids. Sum of sine and cosines = sum of complex exponentials.

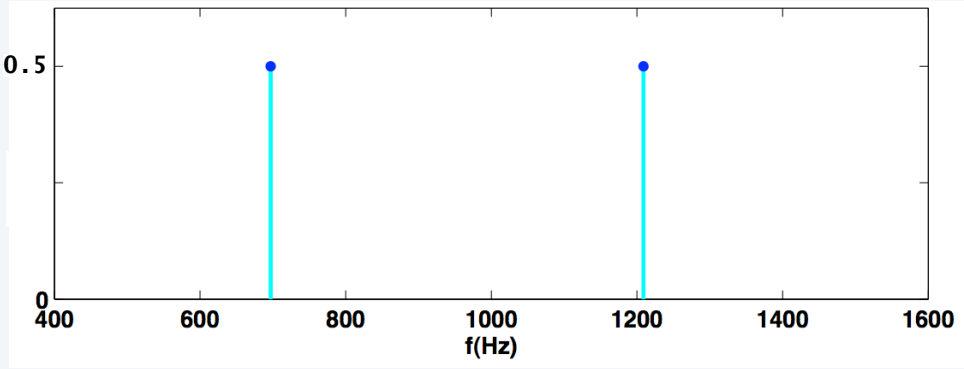
Time domain vs. frequency domain

Signal. [touch tone button 1] $y(t) = \frac{1}{2}\sin(2\pi \cdot 697\ t) + \frac{1}{2}\sin(2\pi \cdot 1209\ t)$

Time domain.



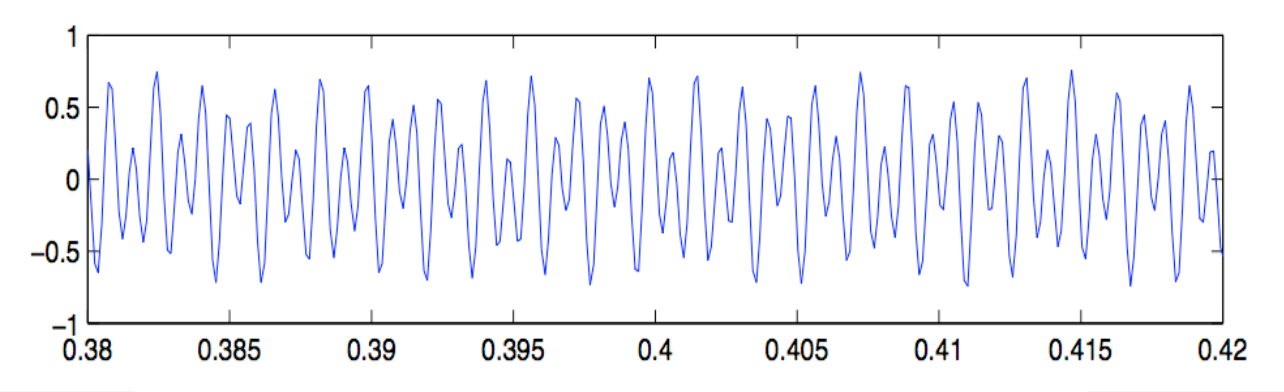
Frequency domain.



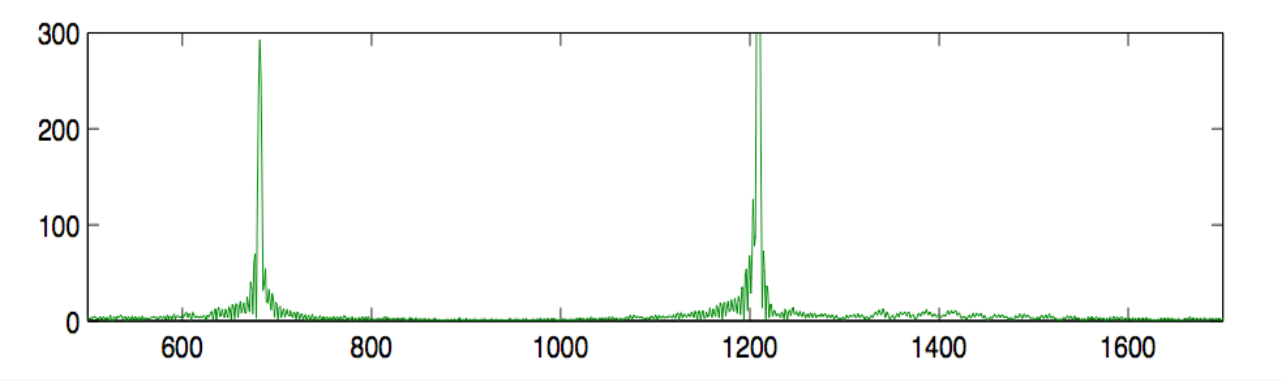
Reference: Cleve Moler, Numerical Computing with MATLAB

Time domain vs. frequency domain

Signal. [recording, 8192 samples per second]



Magnitude of discrete Fourier transform.



Reference: Cleve Moler, Numerical Computing with MATLAB

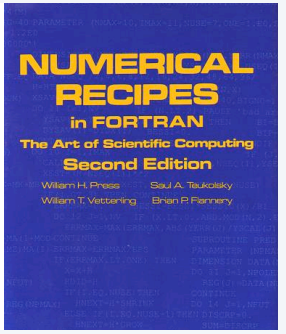
Fast Fourier transform

FFT. Fast way to convert between time-domain and frequency-domain.

Alternate viewpoint. Fast way to multiply and evaluate polynomials.

we take this approach

" If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. " — *Numerical Recipes*



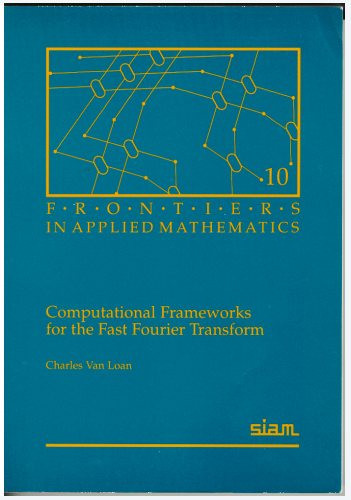
Fast Fourier transform: applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry, ...
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Integer and polynomial multiplication.
- Shor's quantum factoring algorithm.
- ...

" The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. "

— Charles van Loan



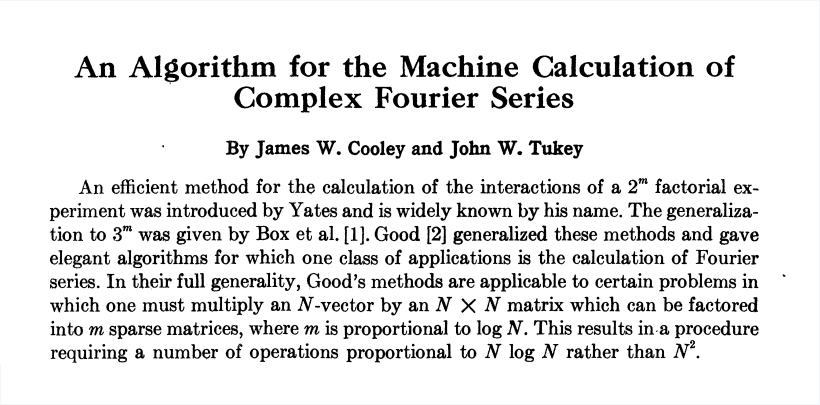
Fast Fourier transform: brief history

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.



An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2.

paper published only after IBM lawyers decided not to
set precedent of patenting numerical algorithms
(conventional wisdom: nobody could make money selling software!)

Importance not fully realized until advent of digital computers.

Polynomials: coefficient representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate. $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

Multiply (convolve). $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

A modest Ph.D. dissertation title

DEMONSTRATIO NOVA
THEOREMATIS
OMNEM FVNCTIONEM ALGEBRAICAM
RATIONALEM INTEGRAM
VNIVS VARIABILIS
IN FACTORES REALES PRIMI VEL SECUNDI GRADVS
RESOLVI POSSE

AVCTORE
CAROLO FRIDERICO GAVSS
HELMSTADII
APVD C. G. FLECKEISEN. 1799

1.

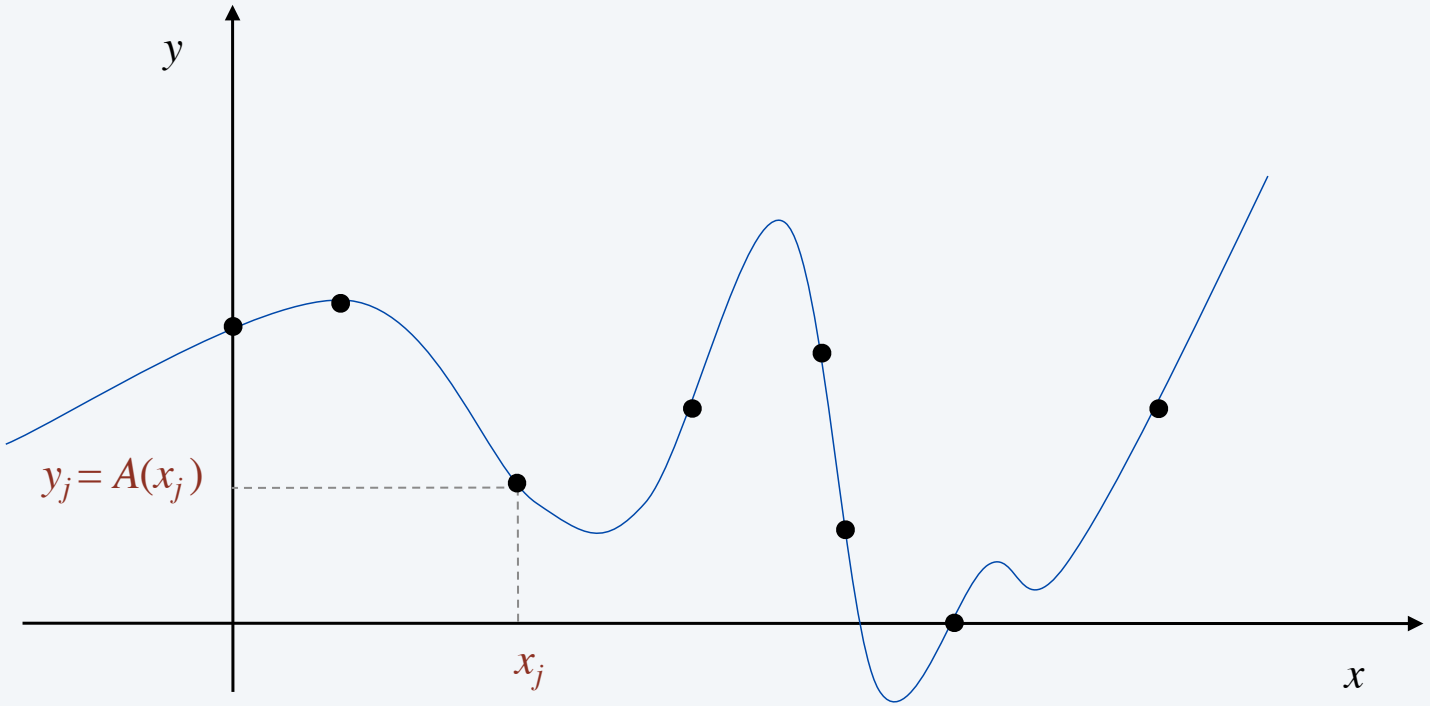
Quaelibet aequatio algebraica determinata reduci potest ad formam $x^m+Ax^{m-1}+Bx^{m-2}+$ etc. $+M=0$, ita vt m sit numerus integer positiuus. Si partem primam huius aequationis per X denotamus, aequationique $X=0$ per plures valores inaequales ipsius x satisfieri supponimus, puta ponendo $x=\alpha$, $x=\beta$, $x=\gamma$ etc. functio X per productum e factoribus $x-\alpha$, $x-\beta$, $x-\gamma$ etc. diuisibilis erit. Vice versa, si productum e pluribus factoribus simplicibus $x-\alpha$, $x-\beta$, $x-\gamma$ etc. functionem X metitur: aequationi $X=0$ satisfiet, aequando ipsam x cuicunque quantitatum α, β, γ etc. Denique si X producto ex m factoribus talibus simplicibus aequalis est (siue omnes diuersi sint, siue quidam ex ipsis identici): alii factores simplices praeter hos functionem X metiri non poterunt. Quamobrem aequatio m^{ti} gradus plures quam m radices habere nequit; simul vero patet, aequationem m^{ti} gradus *pauciores* radices habere posse, etsi X in m factores simplices resolubilis sit:

"New proof of the theorem that every algebraic rational integral function in one variable can be resolved into real factors of the first or the second degree."

Polynomials: point-value representation

Fundamental theorem of algebra. A degree n polynomial with complex coefficients has exactly n complex roots.

Corollary. A degree $n-1$ polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x



Polynomials: point-value representation

Polynomial. [point-value representation]

$$A(x): (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$
$$B(x): (x_0, z_0), \ldots, (x_{n-1}, z_{n-1})$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x): \quad (x_0, y_0 + z_0), \ldots, (x_{n-1}, y_{n-1} + z_{n-1})$$

Multiply (convolve). $O(n)$, but need $2n - 1$ points.

$$A(x) \times B(x): \quad (x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Evaluate. $O(n^2)$ using Lagrange's formula.

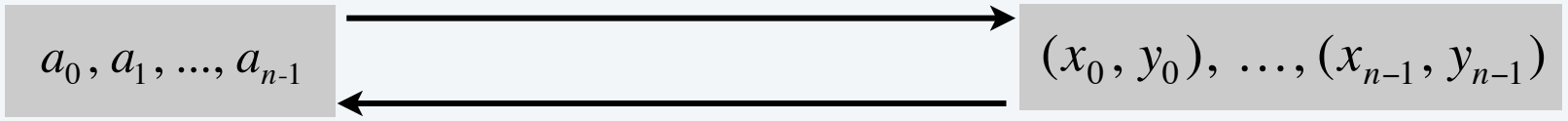
$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting between two representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Goal. Efficient conversion between two representations $\Rightarrow$ all ops fast.



coefficient representation

point-value representation

Converting between two representations: brute force

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^2)$ for matrix-vector multiply (or n Horner's).

Converting between two representations: brute force

Point-value ⇒ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

Running time. $O(n^3)$ for Gaussian elimination.

or $O(n^{2.3727})$ via fast matrix multiplication

Divide-and-conquer

Decimation in frequency. Break up polynomial into low and high powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{low}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$
- $A_{high}(x) = a_4 + a_5 x + a_6 x^2 + a_7 x^3.$
- $A(x) = A_{low}(x) + x^4 A_{high}(x).$

Decimation in time. Break up polynomial into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2).$

Coefficient to point-value representation: intuition

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$. ← we get to choose which ones!

Divide. Break up polynomial into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose two points to be ± 1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$

← **Can evaluate polynomial of degree ≤ n at 2 points by evaluating two polynomials of degree ≤ ½n at 1 point.**

Coefficient to point-value representation: intuition

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- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose four complex points to be $\pm 1, \pm i$.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1).$
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1).$

← **Can evaluate polynomial of degree ≤ n at 4 points by evaluating two polynomials of degree ≤ ½n at 2 point.**

Discrete Fourier transform

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea. Choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

DFT

Fourier matrix F_n

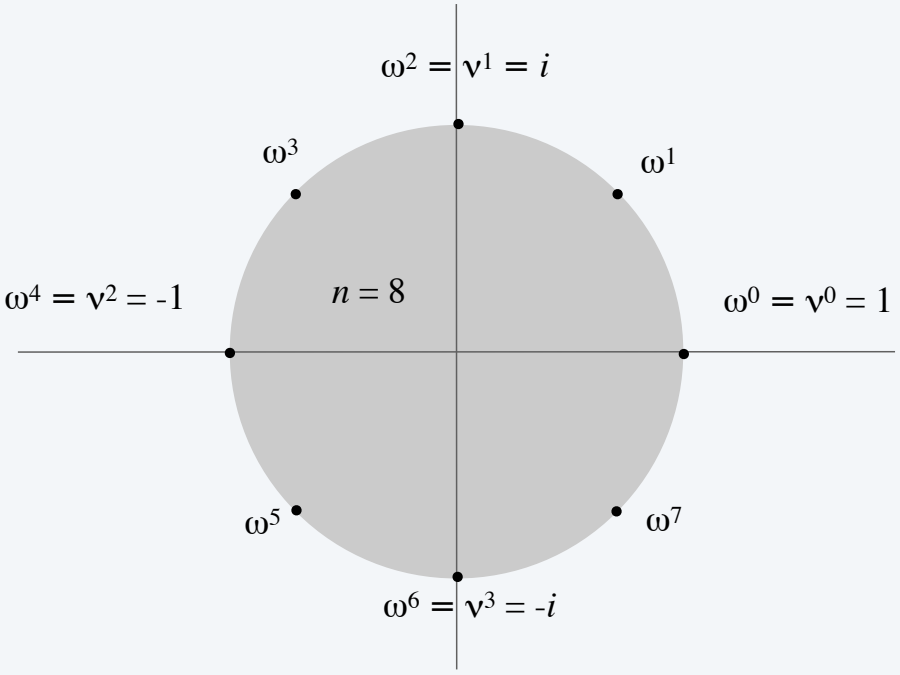
Roots of unity

Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = \omega^2 = e^{4\pi i / n}$.



Fast Fourier transform

Goal. Evaluate a degree $n - 1$ polynomial $A(x) = a_0 + \ldots + a_{n-1}\, x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break up polynomial into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2}\, x^{n/2-1}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1}\, x^{n/2-1}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

$\nu^k = (\omega^k)^2$

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+\frac{1}{2}n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\nu^k = (\omega^{k+\frac{1}{2}n})^2$ $\qquad$ $\omega^{k+\frac{1}{2}n} = -\omega^k$

FFT: implementation

FFT $(n, a_0, a_1, a_2, \ldots, a_{n-1})$

IF $(n = 1)$ RETURN a_0.

$(e_0, e_1, \ldots, e_{n/2-1}) \leftarrow$ FFT $(n/2, a_0, a_2, a_4, \ldots, a_{n-2})$.

$(d_0, d_1, \ldots, d_{n/2-1}) \leftarrow$ FFT $(n/2, a_1, a_3, a_4, \ldots, a_{n-1})$.

FOR $k = 0$ TO $n/2 - 1$.

$\omega^k \leftarrow e^{2\pi i k/n}$.

$y_k \leftarrow e_k + \omega^k d_k$.

$y_{k+n/2} \leftarrow e_k - \omega^k d_k$.

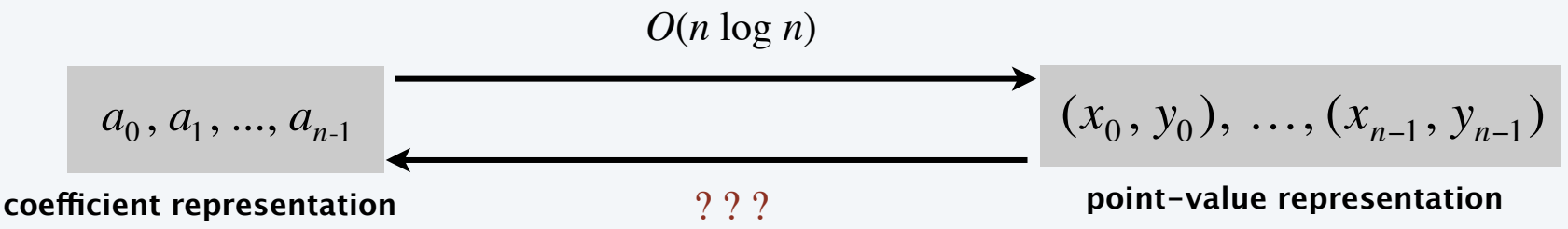
RETURN $(y_0, y_1, y_2, \ldots, y_{n-1})$.

FFT: summary

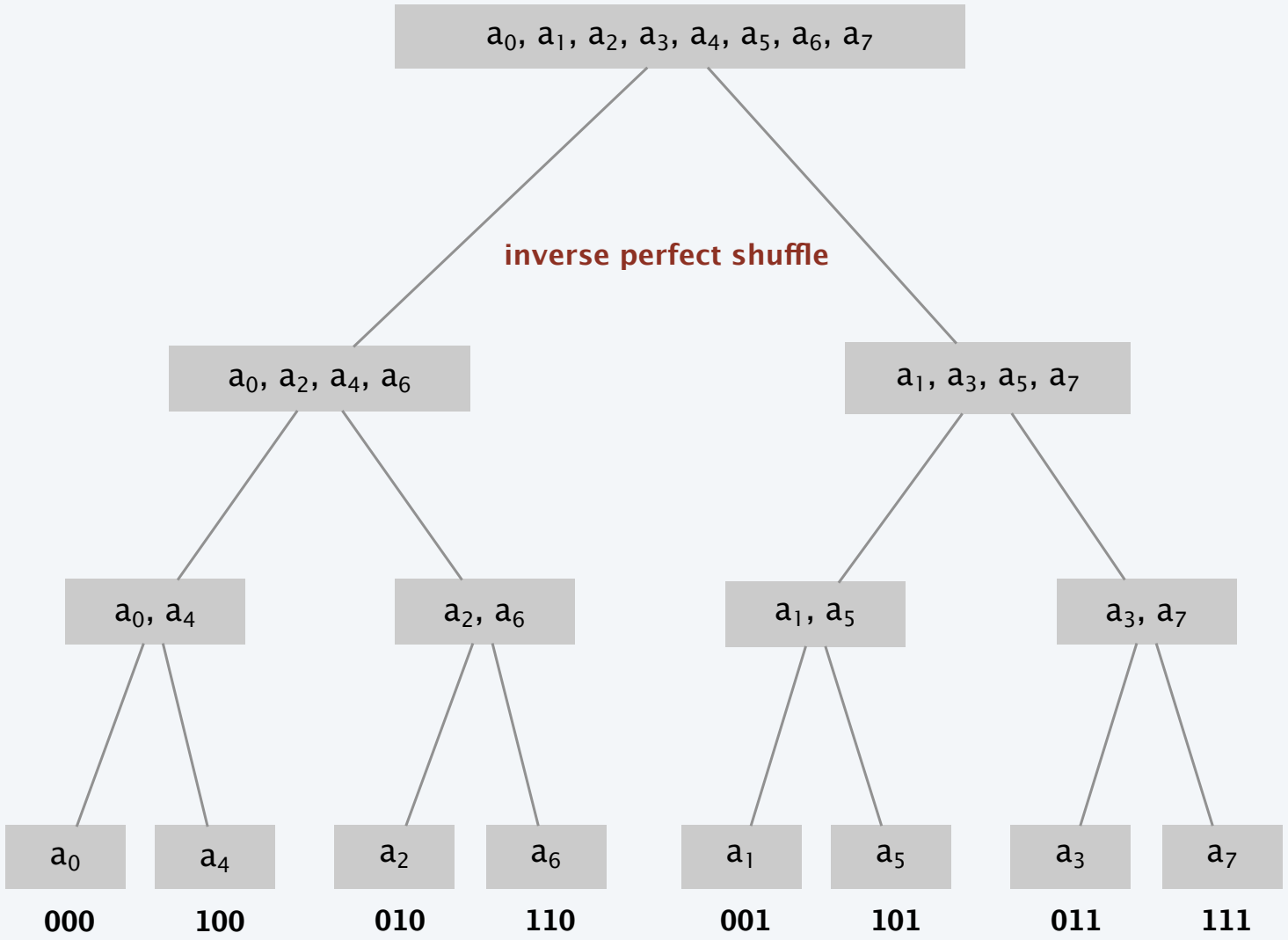
Theorem. The FFT algorithm evaluates a degree $n-1$ polynomial at each of the n^{th} roots of unity in $O(n \log n)$ steps and $O(n)$ extra space.

assumes n is a power of 2

Pf. $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$



FFT: recursion tree



"bit-reversed" order

Inverse discrete Fourier transform

Point-value ⇒ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

Inverse discrete Fourier transform

Claim. Inverse of Fourier matrix F_n is given by following formula:

$$G_n = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

F_n / $\sqrt{n}$ is a unitary matrix

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide the result by n).

Inverse FFT: proof of correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n\, G_n\right)_{k\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{k\,j}\, \omega^{-j\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma (below)

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{k\,j} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \;\Rightarrow$ series sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)\,(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \;\Rightarrow$ series sums to 0. ▪

Inverse FFT: implementation

Note. Need to divide result by n.

INVERSE-FFT $(n, a_0, a_1, a_2, \ldots, a_{n-1})$

IF $(n = 1)$ RETURN a_0.

$(e_0, e_1, \ldots, e_{n/2-1}) \leftarrow$ INVERSE-FFT $(n/2, a_0, a_2, a_4, \ldots, a_{n-2})$.

$(d_0, d_1, \ldots, d_{n/2-1}) \leftarrow$ INVERSE-FFT $(n/2, a_1, a_3, a_4, \ldots, a_{n-1})$.

FOR $k = 0$ TO $n/2 - 1$.

  $\omega^k \leftarrow e^{-2\pi i k/n}$.

  $y_k \leftarrow (e_k + \omega^k d_k)$.

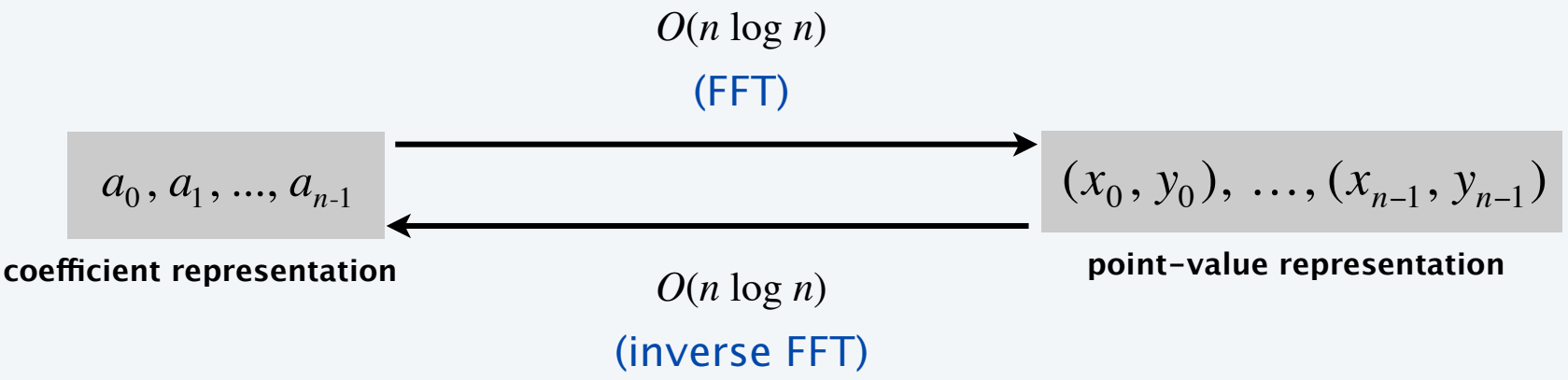
  $y_{k+n/2} \leftarrow (e_k - \omega^k d_k)$.

RETURN $(y_0, y_1, y_2, \ldots, y_{n-1})$.

Inverse FFT: summary

Theorem. The inverse FFT algorithm interpolates a degree $n-1$ polynomial given values at each of the n^{th} roots of unity in $O(n \log n)$ steps.

assumes n is a power of 2

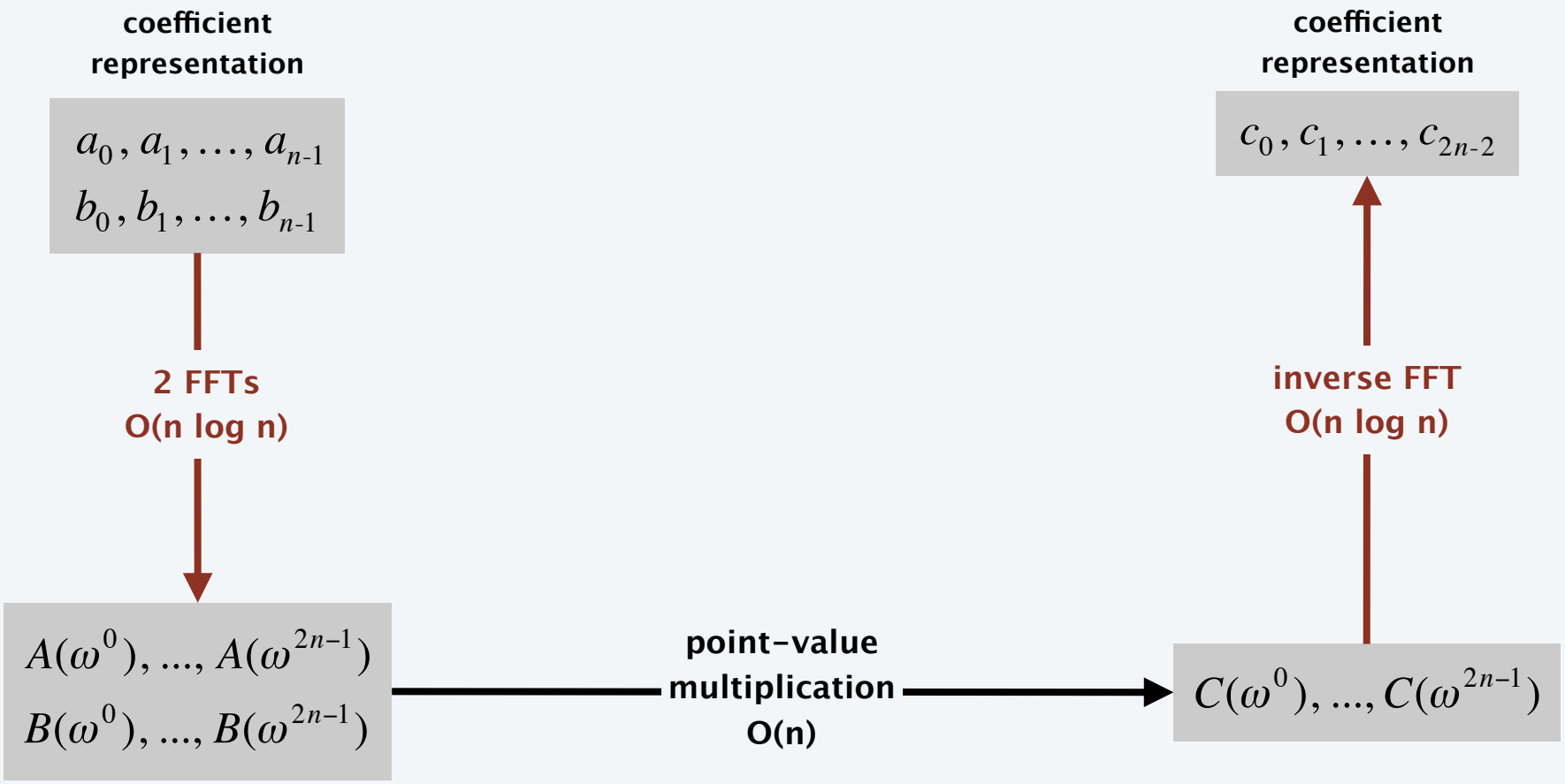


Polynomial multiplication

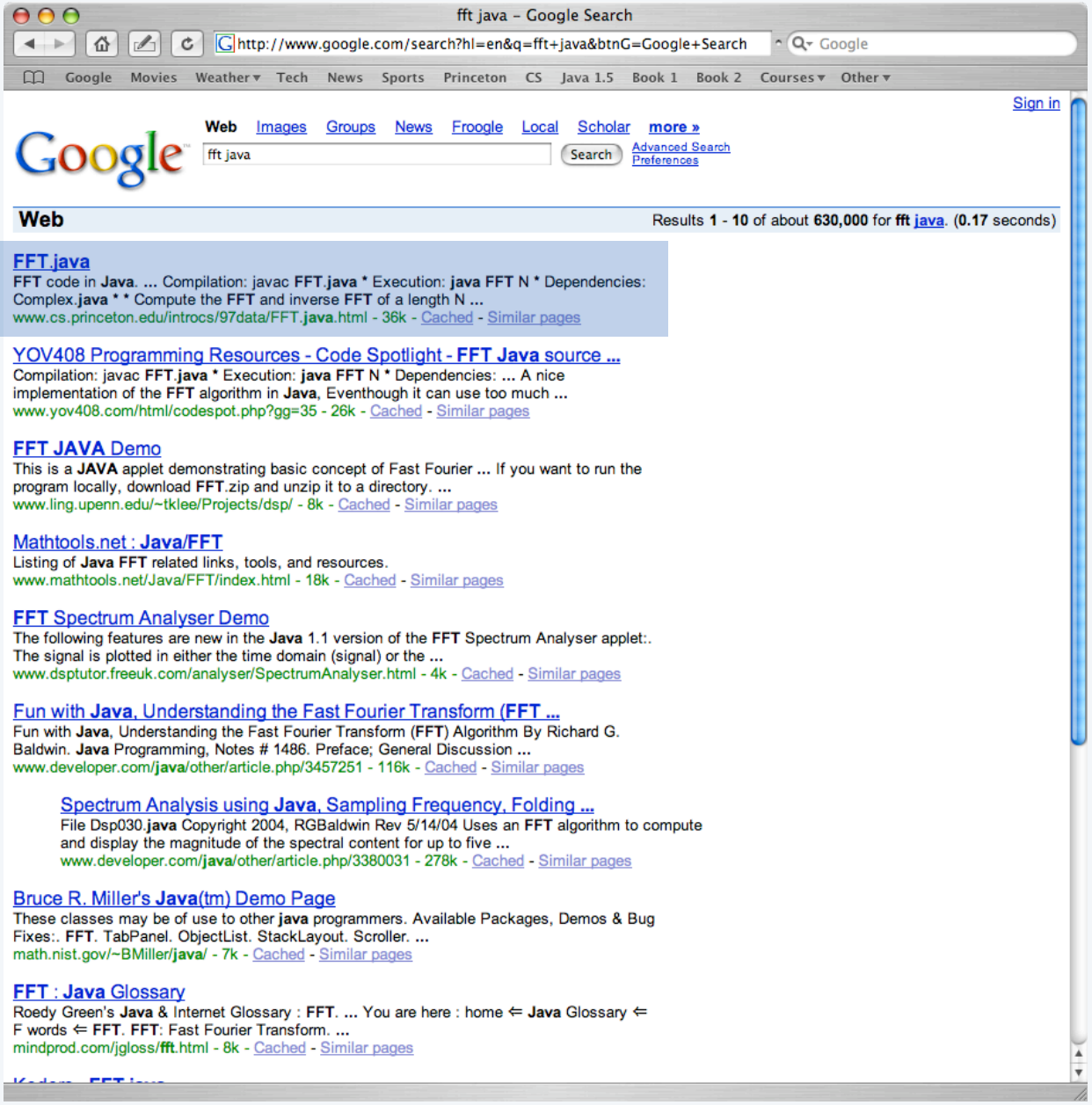
Theorem. Can multiply two degree $n-1$ polynomials in $O(n \log n)$ steps.

Pf.

pad with 0s to make n a power of 2



FFT in practice ?



FFT in practice

Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won 1999 Wilkinson Prize for Numerical Software.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Core algorithm is nonrecursive version of Cooley-Tukey.
- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Runs in $O(n \log n)$ time, even when n is prime.
- Multidimensional FFTs.

http://www.fftw.org

Integer multiplication, redux

Integer multiplication. Given two n-bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product $a \cdot b$.

Convolution algorithm.

- Form two polynomials.
- Note: $a = A(2), b = B(2)$.
- Compute $C(x) = A(x) \cdot B(x)$.
- Evaluate $C(2) = a \cdot b$.
- Running time: $O(n \log n)$ complex arithmetic operations.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Theory. [Schönhage-Strassen 1971] $O(n \log n \log \log n)$ bit operations.

Theory. [Fürer 2007] $n \log n \, 2^{O(\log^* n)}$ bit operations.

Practice. [GNU Multiple Precision Arithmetic Library]

It uses brute force, Karatsuba, and FFT, depending on the size of n.