Chapter 2

Basics of Algorithm Analysis
2.1 Computational Tractability

"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing." - Francis Sullivan
As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - Charles Babbage

Charles Babbage (1864)  
Analytic Engine (schematic)
Polynomial-Time

**Brute force.** For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes $2^N$ time or worse for inputs of size $N$.
- Unacceptable in practice.

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor $C$. 

There exists constants $c > 0$ and $d > 0$ such that on every input of size $N$, its running time is bounded by $cN^d$ steps.

**Def.** An algorithm is poly-time if the above scaling property holds.

$n!$ for stable matching with $n$ men and $n$ women

choose $C = 2^d$
Worst-Case Analysis

**Worst case running time.** Obtain bound on **largest possible** running time of algorithm on input of a given size $N$.

- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

**Average case running time.** Obtain bound on running time of algorithm on **random** input as a function of input size $N$.

- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.
Worst-Case Polynomial-Time

Def. An algorithm is efficient if its running time is polynomial.

Justification: It really works in practice!
- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions.
- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

simplex method
Unix grep
Why It Matters

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>4 sec</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>18 min</td>
<td>$10^{25}$ years</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>$10^{17}$ years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>&lt; 1 sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.
2.2 Asymptotic Order of Growth
Asymptotic Order of Growth

**Upper bounds.** $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \leq c \cdot f(n)$.

**Lower bounds.** $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \geq c \cdot f(n)$.

**Tight bounds.** $T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.

**Ex:** $T(n) = 32n^2 + 17n + 32$.
- $T(n)$ is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
- $T(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$. 
Notation

Slight abuse of notation. $T(n) = O(f(n))$.

- Not transitive:
  - $f(n) = 5n^3; \ g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but $f(n) \neq g(n)$.
- Better notation: $T(n) \in O(f(n))$.

Meaningless statement. Any comparison-based sorting algorithm requires at least $O(n \log n)$ comparisons.

- Statement doesn't "type-check."
- Use $\Omega$ for lower bounds.
Properties

Transitivity.
- If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.
- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$. 
Asymptotic Bounds for Some Common Functions

Polynomials. \(a_0 + a_1 n + \ldots + a_d n^d\) is \(\Theta(n^d)\) if \(a_d > 0\).

Polynomial time. Running time is \(O(n^d)\) for some constant \(d\) independent of the input size \(n\).

Logarithms. \(O(\log_a n) = O(\log_b n)\) for any constants \(a, b > 0\).

\(\uparrow\)

Can avoid specifying the base.

Logarithms. For every \(x > 0\), \(\log n = O(n^x)\).

\(\uparrow\)

\(\log\) grows slower than every polynomial.

Exponentials. For every \(r > 1\) and every \(d > 0\), \(n^d = O(r^n)\).

\(\uparrow\)

every exponential grows faster than every polynomial.
2.4 A Survey of Common Running Times
Linear Time: $O(n)$

Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of $n$ numbers $a_1, \ldots, a_n$.

```
max ← a_1
for i = 2 to n {
    if (a_i > max)
        max ← a_i
}
```
Linear Time: $O(n)$

**Merge.** Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole.

![Diagram of merging two lists]

```
i = 1, j = 1
while (both lists are nonempty) {
    if ($a_i \leq b_j$) append $a_i$ to output list and increment $i$
    else append $b_j$ to output list and increment $j$
}
append remainder of nonempty list to output list
```

**Claim.** Merging two lists of size $n$ takes $O(n)$ time.

**Pf.** After each comparison, the length of output list increases by 1.
O(n log n) Time

**O(n log n) time.** Arises in divide-and-conquer algorithms. Also referred to as linearithmic time.

**Sorting.** Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

**Largest empty interval.** Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

**O(n log n) solution.** Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.
Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of $n$ points in the plane $(x_1, y_1)$, ..., $(x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

\[
\text{min} \leftarrow (x_1 - x_2)^2 + (y_1 - y_2)^2
\]

\[
\text{for } i = 1 \text{ to } n \{ \\
\quad \text{for } j = i+1 \text{ to } n \{ \\
\quad\quad d \leftarrow (x_i - x_j)^2 + (y_i - y_j)^2 \\
\quad\quad \text{if } (d < \text{min}) \\
\quad\quad\quad \text{min} \leftarrow d \\
\quad\} \\
\}\]

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. 

\[\text{don't need to take square roots}\]

\[\text{see chapter 5}\]
Cubic Time: \( O(n^3) \)

Cubic time. Enumerate all triples of elements.

Set disjointness. Given \( n \) sets \( S_1, \ldots, S_n \) each of which is a subset of \( 1, 2, \ldots, n \), is there some pair of these which are disjoint?

\( O(n^3) \) solution. For each pairs of sets, determine if they are disjoint.

```plaintext
foreach set \( S_i \) {
    foreach other set \( S_j \) {
        foreach element \( p \) of \( S_i \) {
            determine whether \( p \) also belongs to \( S_j \)
        }
        if (no element of \( S_i \) belongs to \( S_j \))
            report that \( S_i \) and \( S_j \) are disjoint
    }
}
```
Independent set of size $k$. Given a graph, are there $k$ nodes such that no two are joined by an edge?

$O(n^k)$ solution. Enumerate all subsets of $k$ nodes.

```
foreach subset S of k nodes {
    check whether S in an independent set
    if (S is an independent set)
        report S is an independent set
}
```

- Check whether $S$ is an independent set = $O(k^2)$.
- Number of $k$ element subsets = $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$.

poly-time for $k=17$, but not practical

$k$ is a constant
Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

```
S* ← φ
forall subset S of nodes {
    check whether S is an independent set
    if (S is largest independent set seen so far)
        update S* ← S
}
```