# More Benefits of Adding Sparse Random Links to Wireless Networks: Yet Another Case for Hybrid Networks

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**Abstract.** We theoretically and experimentally analyze the process of adding sparse random links to a random wireless networks modeled as a random geometric graph. While this process has been previously proposed, we are the first to prove theoretical bounds on the improvement to the graph diameter and random walk properties of the resulting graph as a function of the frequency of wires used, where this frequency is diminishingly small. In particular, given a parameter $k$ controlling sparsity, any node has a probability of $\frac{1}{k^2nr^2}$ for being a wired link station. Amongst the wired link stations we consider creating a random 3-regular graph superimposed upon the random wireless network to create model $G_1$, and alternatively we consider a sparser model $G_2$ as well which is a random 1-out graph of the wired links superimposed upon the random wireless network. We prove that the diameter for $G_1$ is $O(k + log(n))$ with high probability, and the diameter for $G_2$ is $O(klog(n))$ with high probability, both of which exponentially improve the $\Theta(\sqrt{\frac{n}{\log n}})$ diameter of the random geometric graph around the connectivity threshold, thus also inducing small-world characteristics as the high clustering remains unchanged. Further, we theoretically demonstrate that as long as $k$ is polylogarithmic in the network size, $G_1$ and $G_2$ have rapidly mixing random walks with high probability, which also exponentially improves upon the mixing time of the purely wireless random geometric graph, which yields direct improvement to the performance of distributed gossip algorithms as well as normalized edge connectivity. Finally, we experimentally confirm that the algebraic connectivities of both $G_1$ and $G_2$ exhibit significant asymptotic improvement over that of the underlying random geometric graph. These results further motivate future hybrid networks and advances in the use of directional antennas.

**Keywords:** small world, wireless networks, mixing time, theory

## 1 Introduction

Ever since the first observation of "six degrees of separation" by Stanley Milgram [1], small-world phenomenon have been noted in numerous diverse network

domains, from the World Wide Web to scientific co-author graphs [2]. The pleasant aspect of the small-world observations is that, despite the high clustering characteristic of relationships with "locality", these various real world networks nonetheless also exhibit short average path lengths as well. This is surprising because purely localized graphs such as low dimensional lattices have very high average path lengths and diameter, whereas purely non-localized graphs such as random edge graph models of Erdos and Renyi [3] exhibit very low clustering coefficient. With intuition consolidating these two extremal graph types, the first theoretical and generative model of small world networks was proposed by Watts and Strogatz [4]: Start with a one dimensional $k$-lattice, and re-wire every edge to a new uniformly at random neighbor with a small constant probability. They showed that even for a very small but constant re-wiring probability, the resulting graph has small average path lengths while still retaining significant clustering.

Despite the prevalence of small world phenomenon in many real-world networks, wireless networks, in particular ad-hoc and sensor networks, do *not* exhibit the small average path lengths required of small-world networks despite the evident locality arising from the connectivity of geographically nearby nodes. Although taking a high enough broadcast radius $r$ clearly can generate a completely connected graph of diameter one, this is a non-realistic scenario because energy and interference also grow with $r$. Rather, from a network design and optimization perspective, one must take the smallest reasonable radius from which routing is still guaranteed. To discuss such a radius in the first place, we must employ a formalization which is common in all theoretical work on wireless networks [5, 6], namely we fix the *random geometric graph* model of wireless networks. Given parameters $n$, the number of nodes, and $r$, the broadcast radius, the random geometric graph $\mathcal{G}(n, r)$ is formed by uniformly at random dispersing the $n$ nodes into the unit square (which is a normalized view of the actual space in which the nodes reside), and then connecting any two nodes iff they are within distance $r$ of each other. Note that due to the normalization of the space, $r$ is naturally viewed as a function of $n$. Given such a model, it is a seminal result of Gupta and Kumar [6] that the *connectivity* property exhibits a sharp threshold for $\mathcal{G}(n, r)$ at critical radius $r_{con} = \sqrt{\frac{\log n}{\pi n}}$, which also corresponds to an average degree of $\log n$. As connectivity is a minimal requirement for routing, $r_{con}$ is the reference point to take for analysis of $\mathcal{G}(n, r)$, and yet, as we shall see, such a radius still yields average path lengths of $\Theta(\sqrt{\frac{\pi n}{\log n}})$ with high probability. Note that a result stated with high probability means probability approaching 1 as $n$ approaches infinity.

This serves as a first motivation for the question: In the spirit of small-world generative models [4] that procured short average path lengths from a geographically defined lattice by adding random "long" edges, can we obtain significant reduction in path lengths by adding random "short cut" wired links to a wireless network? The first to ask this question in the wireless context was Ahmed Helmy [7] who experimentally observed that even using a small amount of wires

(in comparison to network size $n$) that are of length at most a quarter of the physical diameter of the network yields significant average path lengths reduction. Another seminal work on this question is that of Cavalcanti et al. [8] which showed that introducing a fraction $f$ of special nodes equipped with two radios, one for short-range transmission and the other for long-range transmission, improves the connectivity of the network, where this property is seen to exhibit a sharp threshold dependent on both the fraction $f$ and the radius $r$. Other work yet include an optimization approach with a specified sink, in which the placement of wired links is calculated to decrease average path lengths in the resulting topology [9]. The existing body of literature authored by practitioners in the field of wireless networks on inducing small-world characteristics (particularly shortened average path lengths) into wireless networks by either introducing wired links or nodes with special long range radios or directional antennas yields that such hybrid scenarios are eminently reasonable to consider for real networks. Within this small world hybrid wireless networks literature, the closest in spirit to our work, and indeed the only *theoretical* work to our knowledge on hybrid wireless networks is that of [10]. In [10], both deterministic and randomized wiring schemes are given, and bounds are proven on path lengths and energy efficiency under a model in which (i) a designated sink is specified, (ii) routing is based on greedy geographic forwarding only, and (iii) the frequency of wires can be controlled with a parameter $l(n)$. In contrast, in this work, whereas we do allow the wiring frequency to be controlled by a sparsity parameter $k$, we do not assume a designated sink, nor that routing is necessarily greedy geographic forwarding. As such, we obtain very contrasting results to that of [10], in that we find the benefits of totally random wiring, while the totally random wiring exhibited the worst performance under their model and assumptions. Having said this, we now introduce our precise model and assumptions.

In particular, we consider the following models of adding new wired edges: Divide the normalized space into bins of length $k\frac{r}{2\sqrt{2}} \times k\frac{r}{2\sqrt{2}}$, given that the radius is on the order required to guarantee asymptotic connectivity. For each bin, choose a bin-leader. Let the $G_1$ new wiring be such that we form a random cubic graph amongst the bin-leaders and superimpose this upon the random geometric graph. Let the $G_2$ new wiring be such that we form a random 1-out graph amongst the bin-leaders and superimpose this upon the random geometric graph. We prove that the diameter for $G_1$ is $O(k + \log(n))$ with high probability, and the diameter for $G_2$ is $O(k \log(n))$ with high probability, both of which exponentially improve the $\Theta(\sqrt{\frac{n}{\log n}})$ diameter of the random geometric graph, thus also inducing small-world characteristics as the high clustering remains unchanged. Our results on resulting average path lengths are also stable in comparison to using a constant fraction of wire lengths, as that in the work of Ahmed Helmy [7]. To see this note that, for example, using a maximum wire length of one-quarter the maximum distance can be simulated by subdividing the unit square into 16 parts and applying results to the parts separately, then combining into a maximum average path length that is still at most 16 of that within each part.

Whereas the first part of this work concerns bounding the average shortest path lengths of modifications of random geometric graphs, the second part concerns bounding the efficacy of random walks on such graphs. When speaking of a random walk, we connote the *natural* random walk process which is formed by starting from an arbitrary vertex and continuing each step by picking a neighbor uniformly at random from the set of neighbors of the current vertex. If shortest paths may be viewed as optimizing routes under global information, the trace of a random walk can be viewed as a path the node takes under total uncertainty and only local information. Whereas this may not be an optimal source-destination routing method, it can prove useful for general information collection, sampling, gossiping, and discovery of alternate paths when the optimal ones suffer failure[11–14]. The usefulness of random walk based methods depends entirely on the properties of the underlying graph, and can be measured via different metrics dependings on the intent of the method. Two such metrics are the *cover time*, which is the expected time (as in number of steps) in which the random walk visits all nodes of the network, and *mixing time*, which is the maximum time (measured in number of steps starting from an arbitrary node) in which the random walk is within $\epsilon$ distance to the stationary distribution [15].

To give a perspective on what constitutes good cover time properties and what constitutes good mixing time properties, consider that the optimal values for these two properties are exhibited on a clique, and the worst case asymptotic cover time is exhibited on a lollipop graphs and the worst case asymptotic mixing time is exhibited on a barbell graph. The clique has cover time $\Theta(n \log n)$ and constant mixing time, whereas a lollipop graph has $\Theta(n^3)$ cover time, and the barbell graph has polynomial mixing time. Accounting for the degree of a graph, as the clique has maximal degree, for graphs whose degrees are $O(poly \log(n))$, the optimal mixing time is also $\Theta(poly \log(n))$ whereas the optimal cover time remains $\Theta(n \log n)$. Therefore, graphs with poly-logarithmic mixing time are referred to as *rapid mixing* [16, 17]. Previous work [14] showed that whereas the cover time of a random geometric graph about the connectivity threshold is optimal, such graphs far from being rapid mixing. In fact, it was shown that only for radius $r = \Omega(\frac{1}{poly \log n})$, which is exponentially larger than the critical radius required for connectivity $r_{con}$, can the random geometric graph be rapid mixing w.h.p. [18, 19, 14].

In this work, in addition to establishing bounds on resultant path lengths upon sparse random edge additions in the first part, we are the first to consider both theoretically and experimentally the improvement in the resultant mixing time and algebraic connectivity, in comparison to that of the random geometric graph. Although short average path lengths is necessary for a graph to exhibit optimal random walk sampling properties, it is far from sufficient (the barbell graph being a notable counterexample). As a strange omission, the small world literature thus far has primarily ignored spectral gap as a measure in their analyses and generative models despite the known expansion of random edge graph models [20, 21]. It is well established that the mixing time is intrinsically related

to the node expansion, edge expansion [22], algebraic connectivity, and random walk properties of the given graph [16, 17, 15].

Our motivation is as follows: Yet another limitation of random geometric graphs in comparison to random edge graph models that is especially problematic for oblivious routing, sampling, and gossiping applications[11–13] is that, whereas sparse random regular graphs as well as random connected Erdos-Renyi graphs are expanders with excellent mixing properties, connected random geometric graphs $\mathcal{G}(n, \Theta(r_{con}))$ are far from being rapidly mixing. In general, additional edges need not improve the mixing time of the resulting graph. Fortunately, in this work, we are able to show that sparse additional edge additions, when done randomly as in models $G_1$ and $G_2$, do indeed yield exponentially improved mixing time. We show these results for $G_1$ and $G_2$ using a *conductance argument* and confirm with experimental calculation of the resultant spectral gap of the normalized Laplacian, which is a normalized measure of algebraic connectivity [23]. More recently, algebraic connectivity has been noted by network scientists to be an intrinsic measure of the robustness of a complex network to node and link failures [24], thus giving even stronger motivation for our present study.

In terms of related work, we must note the work [25] of Abraham Flaxman, which is an excellent related work in which the spectra of randomly *perturbed* graphs have been considered in a generality that already encompasses major small-world models thus far. [25] demonstrates that, no matter what is the starting graph $G_0$, adding random 1-out edges at every node of $G_0$ will result in a graph with constant spectral gap (the best possible asymptotically). The work also presents a condition in which a random Erdos-Renyi graph superimposed upon the nodes of $G_0$ would yield good expansion, whereas without that condition the resulting graph may have poor expansion. Despite the apparent generality of the work in terms of the arbitrariness of the underlying graph that is considered, unfortunately the results do not generalize to situations in which not all *edges* are involved in a wired linkage. Notably, the small world models thus far also require such a high probability of new random links. In contrast, in this work, we focus on adding sparse random wires and presenting general bounds on mixing time dependent on the frequency of wired link stations. In particular, the fraction of nodes involved in a wired link will be no more than $O(\frac{1}{\log n})$, and in general shall be $O(\frac{1}{k^2 \log n})$, both of which are asymptotically diminishing fractions.

Finally, we note that this work is a significant extension to the author's conference paper [26].

## 2 Theoretical Preliminaries

The results can be divided logically into those concerning average path lengths and those concerning random walk sampling properties. Therefore, the preliminaries are also so divided.

### 2.1 Random Geometric Graph Preliminaries

Random geometric graphs above the connectivity threshold exhibit certain "smooth lattice-like" properties including uniformity of node distribution and regularity of node degree, that are useful in their analysis. As introduced in [14], we utilize the notion of a *geo-dense* graph to characterize such properties, that is, a geometric graph (random or deterministic) with uniform node density across the unit square. It was shown that *random* geometric graphs are *geo-dense* and for radius $r_{\text{reg}} = \Theta(r_{\text{con}})$ all nodes have the same order degree[14]. We formally present the relevant results from [14] in this section, as well as the notion of *bins*, namely the equal size areas that partition the unit square. Such "bins" are the concrete link between lattices and random geometric graphs, essentially forming the lattice backbone of such graphs.

Formally, a **geometric graph** is a graph $G(n, r) = (V, E)$ with $n = |V|$ such that the nodes of $V$ are embedded into the unit square with the property that $e = (u, v) \in E$ if and only if $d(u, v) \leq r$ (where $d(u, v)$ is the Euclidean distance between points $u$ and $v$). In wireless networks, $r$ naturally corresponds to the broadcast radius of each node. The following formalizes geo-denseness for geometric graphs:

**Definition 21** *[14] Let $G(n, r(n))$ be a geometric graphs [1]. For a constant $\mu \geq 1$ we say that such a class is $\mu$-**geo-dense** if every square bin of size $A \geq r^2/\mu$ (in the unit square) has $\Theta(nA)$ nodes.*[2]

The following states the almost *regularity* of geo-dense geometric graphs[14]:

**Lemma 22** *Let $G(n, r)$ be a 2-geo-dense geometric graph with $V$ the set of nodes and $E$ the set of edges. Let $\delta(v)$ denote the degree (i.e number of neighbors) of $v \in V$. Then: (i) $\forall v \in V \ \ \delta(v) = \Theta(nr^2)$ and (ii) $m = |E| = \Theta(n^2 r^2)$*

Recall that the critical radius for connectivity $r_{\text{con}}$ is s.t $\pi r_{\text{con}}^2 = \frac{\log n}{n}$[6]. The following is the relevant lemma that states that *random* geometric graphs with radius at least on the order of that required for connectivity are indeed geo-dense[14]:

**Lemma 23 (Geo-density of $\mathcal{G}(n, r)$)** *For constants $c > 1$ and $\mu \geq 2$, if $r^2 = \frac{c\mu \log n}{n}$ then w.h.p. $\mathcal{G}(n, r)$ is $\mu$-*geo-dense*. That is, w.h.p. (i) any bin area of size $r^2/\mu$ in $\mathcal{G}(n, r)$ has $\Theta(\log n)$ nodes, and (ii) $\forall v \in \mathcal{G}(n, r)$, $\delta(v) = \Theta(nr^2)$ and $m = |E| = \Theta(n^2 r^2)$. Further, note that increasing the radius $r$ can only smoothen the distribution further, maintaining regularity.*

Both geo-denseness and other results both in this paper and on previous work on random geometric graphs follows from a "folk theorem" often referred to as Coupon Collection (due to the example process given), so we state this before continuing the characterization of random geometric graphs [15]:

[1] either random or deterministic

[2] Note that if a geometric graph is $\mu_1$-geo-dense then it is also $\mu_2$-geo-dense for any $\mu_2 < \mu_1$. That is, if a distribution is smooth for some granularity $x$ it can only be smoother for a coarser granularity $y$.

**Theorem 24 (Coupon Collection)** *Assume that there are a total of $n$ types of coupons, and one attempts to collect all types by picking $m$ coupons independently and uniformly at random. Upon this process, let $x_i$ denote the number of coupons of type $i$ that have been collected. Then if $m = \Omega(n \log n)$, for any types $i$ and $j$, $x_i = \Theta(x_j)$ with high probability. In particular, the probability concerned is* very high *as $1 - O(\frac{1}{n})$.*

This process and the corresponding folk theorem is also alternatively referred to as "Balls in Bins".

Given that geo-denseness of connected random geometric graphs is established, we wish to utilize the "binning" directly for its lattice-like properties. As such, for the sake of notational convenience, we shall introduce the notion of a *lattice skeleton* for geo-dense geometric graphs, including random geometric graphs above connectivity:

**Definition 25** *Let $G(n, r(n))$ be a $\mu$-geo-dense geometric graph for some constant $\mu \geq 1$. Define the* **$\mu$-lattice skeleton** *for $G(n, r)$ to be $LS(G(n, r)) = (L, B_{i,j})$ such that $L = (B, E_B)$ is the $\frac{\sqrt{\mu}}{r} \times \frac{\sqrt{\mu}}{r}$ two-dimensional lattice on $\frac{\mu}{r^2}$ bin-points, where each vertex set $B_{i,j}$ represents the set of nodes of $G(n, r)$ that lie in lattice location $\{i, j\}$ of $L$. Further, for a node $v \in G(n, r)$ with Cartesian coordinates $(x, y) \in [0, 1]^2$, denote by $B(v)$ the lattice-bin containing $v$, namely the bin $B_{i,j}$ such that $i = ceiling(\frac{x\sqrt{\mu}}{r})$ and $j = ceiling(\frac{y\sqrt{\mu}}{r})$.*

We note, before proceeding, that such ideas of geometric bins in representing random geometric graphs are not new to this work (hence, they appear as preliminaries), but rather have arisen naturally in a number of theoretical works on wireless networks above the connectivity regime. The idea of the random geometric graph as a global lattice skeleton composed with local cliques in particular as appears here has been formalized via the global-local decomposition representation of such graphs introduced the author's thesis [27].

What is directly clear by geo-denseness is that there is not much variance in the sizes of the bins:

**Remark 26** *Let $LS(G(n, r)) = (L, B_{i,j})$ be the $\mu$-lattice skeleton of a $\mu$-geo-dense geometric graph $G(n, r)$. Then, $\forall i, j, |B_{i,j}| = \Theta(nr^2)$.*

Further, utilizing the choice of $\mu \geq 5$, we may make the stronger statement that the connectivity of the lattice is inherited in the nodes of the overall graph. The justification is simply that $r$ becomes the length of the diagonal connecting the farthest points of adjacent bins, and we formalize combining with Remark 26 and Lemma 23:

**Lemma 27** *Let $LS(G(n, r)) = (L, B_{i,j})$ be the 5-lattice skeleton of a 5-geo-dense geometric graph $G(n, r)$. Then $\forall i, j, k, l$ if $((i, j), (k, l)) \in L$ then $\forall v \in B_{i,j}, u \in B_{k,l}, (v, u) \in G(n, r)$*
*In particular, for $c > 1$ if $r^2 \geq \frac{5c}{\pi} r_{con}^2$ then the 5-lattice skeleton $LS(\mathcal{G}(n, r) = (L, B_{i,j})$ of random geometric graph $\mathcal{G}(n, r)$ satisfies the following:*

*(i)* $\forall i,j |B_{i,j}| = \Theta(nr^2)$ *w.h.p.,*
*(ii)* $\forall i,j$ *if* $v,u \in B_{i,j}$ *then* $(v,u) \in \mathcal{G}(n,r)$,
*(iii)* $\forall i,j,k,l$ *if link* $((i,j),(k,l)) \in L$ *then* $\forall v \in B_{i,j}, \forall u \in B_{k,l}, (v,u) \in \mathcal{G}(n,r)$, *and*
*(iv)* $\forall i,j,k,l$ *if* $\min\{|i-k|,|j-l|\} \geq 4$ *then* $\forall v \in B_{i,j}, \forall u \in B_{k,l}, (v,u) \notin \mathcal{G}(n,r)$

From (ii) it is clear that each bin $B_{i,j}$ forms a clique (namely all pairs of nodes within are connected directly by length one paths). From (iii) it follows that a path in the lattice $L$ yields a path in the graph $\mathcal{G}(n,r)$ as well, while (iv) bounds the converse situation in that nodes that lie in bins at least 4 lattice-hops away cannot be directly connected in the graph $\mathcal{G}(n,r)$ either. In particular, (iii) and (iv) yield that pairwise distances between points in the graph $\mathcal{G}(n,r)$ inherit the shortest paths (Manhattan) distances in the corresponding lattice-bins of the lattice-skeleton, up to constant factors. We formalize with the following corollary:

**Corollary 28** *For* $c > 1$ *if* $r^2 \geq \frac{5c}{\pi} r_{con}^2$ *then the* 5*-lattice skeleton* $LS(\mathcal{G}(n,r)) = (L, B_{i,j})$ *of random geometric graph* $\mathcal{G}(n,r)$ *satisfies the following w.h.p.:* $\forall u,v \in \mathcal{G}(n,r), dist_{\mathcal{G}(n,r)}(u,v) = \Theta(dist_L(B(u),B(v))+1)$ *where the function* $dist_G$ *indicates shortest paths distances in graph* $G$.

Having established that connectivity and distances for $\mathcal{G}(n,r)$ with radius at least a small constant times $r_{con}$ roughly preserve connectivity and distances in the $\frac{\sqrt{\mu}}{r} \times \frac{\sqrt{\mu}}{r}$ lattice skeleton, let us then consider the number of lattice-nodes $N_{d,L}(v)$ that are at lattice distance exactly $d$ away from $v$ in the lattice $L$: Clearly, $N_{d,L}(v)$ grows linearly in $d$ by a simple induction on upper and lower bounds. And, the maximum distance to consider is $d = \Theta(\frac{\mu}{r^2})$. Moreover, due to the smooth distribution of random geometric graph nodes in the lattice bins, we must have that the *fraction* $f_{d,L}(B(v))$ of lattice bins at lattice-distance exactly $d$ away from $B(v)$ must be on the same order as the *fraction* of random geometric graph nodes $f_{d,\mathcal{G}(n,r)}(v)$ at hop-distance exactly $d$ away from $v$. Thus, $f_{d,\mathcal{G}(n,r)}(v) = \Theta(f_{d,L}(B(v))) = \Theta(\frac{N_{d,L}(v)}{|L|}) = \Theta(\frac{N_{d,L}(v) r^2}{\mu}) = \Theta(dr^2)$

Such fractions represent the probability that a node is at distance $d$ away from a given node $v$. Thus, we may calculate the average path length $APL$ which is the expectation of that probability distribution on the very function $d$ itself:
$APL(\mathcal{G}(n,r)) = \sum_{d=1}^{D(\mathcal{G}(n,r))} d f_{d,\mathcal{G}(n,r)}(v) = \Theta(\sum_{d=1}^{2\frac{\sqrt{\mu}}{r}} d f_{d,L}(B(v))) = \Theta(\sum_{d=1}^{2\frac{\sqrt{\mu}}{r}} d^2 r^2) = \Theta(r^2 \sum_{d=1}^{2\frac{\sqrt{\mu}}{r}} d^2) = \Theta(r^2 \frac{1}{r^3}) = \Theta(\frac{1}{r})$

Thus, the average path length for random geometric graphs above the connectivity threshold is the same as the order of the diameter (maximum shortest path lengths) for such graphs, which is $\Theta(\frac{1}{r})$. While the dependence on the radius $r$ in that term may seem optimistic at first, noting that $r$ should be kept as low as possible to reduce energy overhead and interference of the ad-hoc network represented, a realistic constraint on $r$ becomes $r = \Theta(r_{con}) = \Theta(\sqrt{\frac{\log n}{n}})$,

namely that achieving degree $\Theta(\log n)$. Thus, $APL$ of reasonable random geometric graphs (of minimal radius guaranteeing connectivity) scales quite badly as $\Theta(\sqrt{\frac{n}{\log n}})$.

### 2.2 Random Walk and Connectivity Preliminaries

When speaking of a random walk, we connote in particular this natural process: If the random walk is currently at node $q$, then the simplest probabilistic rule by which to choose the next node is simply to choose a node uniformly at random from among the set of neighbors of $q$. And, the Markov chain $\mathfrak{M} = (\Omega, P)$ corresponding to such a random walk on a graph $G = (V, E)$ is the *simple random walk* on $G$. For such $G$, for any node $v \in V$, let $\delta(v)$ denote the degree of $v$, that is the number of neighbors of $v$ in $G$ and let $P(v, u) = \frac{1}{\delta(v)}$ for $(v, u) \in E$ and 0 otherwise. In linear algebraic terms, the process is an application of $P$ to the current distribution vector $v_t$ of step $t$, where the initial distibution vector $v_0$ is concentrated completely at an arbitrary node: $v_t = v_{t-1}P = v_0P^t$.

In such terms, the *stationary distribution* of $\mathfrak{M}$, if such exists, is the unique probability vector $\pi$ such that

$$\pi P = \pi \tag{1}$$

The stationary distribution being a fixed point vector that remains unchanged upon operator $P$ is also the distribution to which the random walk converges, regardless of the starting point, given that $G$ is connected and non-bi-partite (which is guaranteed by any odd length cycle):

$$\pi = \lim_{t \to \infty} v_0 P^t, \forall v_0 \tag{2}$$

Moreover, when the underlying graph $G$ is regular, then the stationary distribution is the uniform distribution [28], and this statement remain true asymptotically when $G$ is almost-regular as well (namely, when the degree of every node is $\Theta(f(n))$ for the same function $f$). Therefore, for almost regular graphs, it is clear that the random walk samples efficiently at stationarity, and the faster the random walk on a regular graph converges to stationarity, the greater its load-balancing qualities. This rate of convergence to stationarity is called the *mixing time*.

To define mixing time, we must first introduce the relevant notion of distance over time. Let $x$ be the state at time $t = 0$ and denote by $P^t(x, \cdot)$ the distribution of the states at time $t$. The *variation distance* at time $t$ with respect to the initial state $x$ is defined to be [16]

$$\Delta_x(t) = \max_{S \subseteq \Omega} |P^t(x, S) - \pi(S)|$$

Note that when the state space $\Omega$ is finite it can be verified that [14]:

$$\Delta_x(t) = \frac{1}{2} \sum_{y \in \Omega} |P^t(x, y) - \pi(y)|$$

Now we may formally define the mixing time as the following function [16]

$$\tau_x(\epsilon) = min\{t \mid \Delta_x(t') \leq \epsilon, \forall t' \geq t\}$$

A chain $\mathfrak{M}$ is considered *rapidly mixing* iff $\tau_x(\epsilon)$ is $O(poly(\log(n/\epsilon)))$. Clearly, as the name indicates, for a random walk to be used for efficient sampling (according to its stationary distribution), it should be rapidly mixing.

Now, on the way towards proving the rapid mixing property of a random walk, we shall make use of a number of beautiful connections amongst mixing time, the eigenvalues of the Markov chain (in particular the *spectral gap*, namely the difference between the first and second eigenvalues), and connectivity properties of the underlying graph as encapsulated by notions called *conductance* which is a normalized form of *expansion*. In introducing the connection between expansion and rapid mixing, we note that intuitively graphs with minimal "bottlenecks" have also a lower the probability of getting stuck in any particular set of states, and thus a faster mixing time as well. We shall see that the graph-connectivity based property of "no bottlenecks" is formalized in a continuous manner with the notion of *conductance* and in a combinatorial manner with *expansion*. And, then we shall make the relationship between conductance and mixing time precise.

In fact, one of the motivations we have in considering random edge additions to random geometric graphs is precisely based on the nice connectivity properties that random $d$-regular graphs possess, which we shall see are very much not possessed by random geometric graphs: Random $d$-regular graphs are *expanders* w.h.p. for $d \geq 3$.[20, 21]. The combinatorial meaning of this statement is as follows: W.h.p., every subset $S \subset V$ has many edges separating $Cut(S, \bar{S})$, particularly $|Cut(S, \bar{S})| = \alpha d|S|$ for a constant $\alpha > 0$ [15]. In general, the *expansion* of a graph is thus the ratio of the worst case cut divided by the size of the set, and an expander is a graph with constant expansion. Note that the property of a graph being an expander is a much stronger notion than $k$-connectivity in that it clearly implies an edge connectivity that is at least on the same asymptotic order as the minimum degree, but it further requires that the density of edges separating any set from the rest of the graph is *proportional* to the size of the set. In fact, being an expander is an extremal property and also much stronger than both the properties of having logarithmic diameter and being rapidly mixing. As such, unsurprisingly, we will not be able to prove that our graphs resulting from random edge additions are expanders. Nonetheless, we will be able to prove *sufficient* expansion so as to guarantee that the random walk is rapid mixing. We will do so by bounding the conductance.

The *conductance* of a reversible Markov chain $\mathfrak{M}$ is defined by [17]

$$\Phi = \Phi(\mathfrak{M}) = \min_{S \subset \Omega, 0 < \pi(S) \leq 1/2} \frac{Q(S, \overline{S})}{\pi(S)}$$

where $\overline{S} = \Omega - S$, $\pi(S)$ is the probability density of $S$ under the stationary distribution $\pi$, and $Q(S, \overline{S})$ is the sum of $Q(v, u) = \pi(v)P(v, u)$ over all $(v, u) \in S \times \overline{S}$.

In graph-theoretic terms, the conductance of $\mathfrak{M}$ is the minimum over all subsets $S \subset \Omega$ of the ratio of the weighted flow across the cut $Cut(S, \overline{S})$ to the weighted capacity of $S$, and as such is clearly a continuous measure of "the degree of no bottlenecks" property. For almost regular graphs of degree $\Theta(d)$, we may simplify the expression for conductance as follows:

$$\phi(G) = \min_{S \subset V} \frac{|Cut(S, \bar{S})|}{d|S|} \tag{3}$$

And, for this case, it is clear that conductance is a type of normalized measure of expansion where the degree is taken into account as well. Now that we have defined expansion and conductance, we must soon relate these measures to the rapid mixing property. We do this by connecting both conductance and mixing time to the *spectral gap*.

As the stationary distribution $\pi$ is defined to be such that $\pi P = \pi$, it corresponds to the eigenvalue $\lambda_0 = 1$ of $P$. Let the rest of the eigenvalues of $P$ in decreasing order of absolute value be: $1 = \lambda_0 \geq |\lambda_1| \geq \cdots \geq |\lambda_{n-1}| \geq -1$. For a finite, connected, non-bipartite Markov chain as the type in this work, the rate of convergence to $\pi$, which as you may recall is captured by the mixing time, is governed by the difference between the first and second eigenvalues, namely the *spectral gap* which is $1 - \lambda_1$ [16]. And, here are the theorems establishing these relationships:

**Theorem 29** *For an ergodic Markov chain*[3]*, the quantity $\tau_x(\epsilon)$ satisfies*

*i* $\tau_x(\epsilon) \leq (1 - \lambda_{max})^{-1}(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$

*ii* $\max_{x \in \Omega} \tau_x(\epsilon) \geq \frac{1}{2}\lambda_{max}(1 - \lambda_{max})^{-1} \ln(2\epsilon)^{-1}$

And, to relate conductance explicitly to mixing time, it thus suffices to bound the spectral gap with the conductance:

**Theorem 210 ([16])** *The second eigenvalue $\lambda_1$ of a reversible Markov chain $\mathfrak{M}$ satisfies*

$$1 - 2\Phi \leq \lambda_1 \leq 1 - \frac{\Phi^2}{2}$$

Combining, as in [14]:

**Corollary 211 ([17])** *Let $\mathfrak{M}$ be a finite, reversible, ergodic Markov chain with loop probabilities $P(x, x) \geq \frac{1}{2}$ for all states $x$. Let $\Phi$ be the conductance of $\mathfrak{M}$. Then, for any initial state $x$, the mixing time of $\mathfrak{M}$ satisfies*

$$\tau_x(\epsilon) \leq 2\Phi^{-2}(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$$

In particular, the following is immediate too:

[3] Ergodicity is guaranteed by the chain being finite, connected, and non-bipartite, as we have in this work.

*Remark 1.* For a random walk to be rapid mixing, it is necessary and sufficent that the conductance be inverse poly-logarithmic.

Finally, we must speak of the mixing properties of the random geometric graph above the connectivity threshold as shown in [18, 19, 14]:

**Theorem 212** *Given* $r = \Omega(r_{con})$, $\phi(\mathcal{G}(n, \Theta(r))) = \Theta(r)$. *Similarly,* $\lambda_2(\mathcal{G}(n, r)))$ *is* $\Omega(r^2)$ *and* $O(r)$.

In particular at $\Theta(r_{con})$, $\lambda_2(\mathcal{G}(n, r))$ is $\Omega(\frac{\log n}{n})$ and $O(\sqrt{(}\frac{\log n}{n})$. This gives a mixing time of $\Omega(n^{\epsilon})$ for $\epsilon > 0$. Compare to a *rapid mixing* Markov chain which requires only $O(polylog(n))$ steps: Possible in wireless network only for very large radius $r = \Theta(\frac{1}{polylog(n)})$, exponentially larger than $r_{con}$. Re-stating, from [14]:

**Corollary 213 (Mixing Time of RGG)** *Radius* $r = \Omega(1/poly(\log n))$ *is w.h.p. necessary and sufficient for* $\mathcal{G}(n, r)$ *to be rapidly mixing.*

On the other hand, recall: Even sparse random regular graphs are rapid mixing:

**Remark 214** *It is well-known that the random 3-regular graph* $G_{R,1}(k)$ *is an expander with high probability [22, 20]. Therefore,* $G_{R,1}(k)$ *also exhibits diameter and average path lengths asymptotically at most logarithmic in its vertex set* $|V_R(k)|$, *with high probability.*

## 3 Models of Random Edge Additions

As we start to consider the business of adding random edges to a given initial graph $G_0 = (V_0, E_0)$, note that the set of additional edges $E_R$ and the existing nodes connected by them $V_R \subset V$ forms a graph $G_R$ such that the resulting graph $G = (V, E) = G_0 + G_R$ has $V = V_0 \geq V_R$ and $E = E_0 + E_R$.[4]. That is, it is also convenient to view the additional random edges as a new graph $G_R$ superimposed upon the original graph $G_0$.

Given such a characterization, let us be given a 5-geo-dense geometric graph $G_0 = (V_0, E_0) = G(n, r)$ with 5-lattice-skeleton $(L, B_{i,j})$. In particular, note from Lemma 23 that results apply to any $G_0 = \mathcal{G}(n, \Omega(r_{con}))$. Given parameter $k \geq 1$, let vertex set $V_R(k)$ be generated as follows: For any $i, j \leq \frac{\sqrt{5}}{\sqrt{k}r}$ pick a node $v_{i,j}$ uniformly at random from the nodes in the set of bins $\mathcal{B}_{i,j} = \cup_{ki \leq i' \leq (k+1)i, kj \leq j' \leq (k+1)j} B_{i',j'}$[5], and set $v_{i,j} \in V_R(k)$. For the case $k = 0$, let $V_R(0) = V_0$. We now define the various types of random edge sets $E_{R,i}$ for graphs $G_{R,i}(k) = (V_R(k), E_{R,i})$ whose superimpositions upon $G_0$ we shall consider in this work: Let $E_{R,1}$ be generated as follows: For every node $v \in V_R(k)$ pick 3

[4] Such a notation is also consistent with [25]

[5] This bin union is simply the formalization of a single contiguous bin $k \times k$ as large as the original bins.

neighbors in $V_R(k)$ uniformly at random, discarding situations in which any node has degree greater than 3. Thus, the resulting graph $G_{R,1}(k) = (V_R(k), E_{R,1})$ is the random 3-regular Erdos-Renyi graph defined on vertex set $V_R(k)$. Let $E_{R,2}$ be generated as follows: For every node $v \in V_R(k)$ pick 1 neighbor in $V_R$ uniformly at random. Thus, the resulting graph $G_{R,2}(k) = (V_R(k), E_{R,2})$ is the random 1-out graph defined on vertex set $V_R(k)$.

Similarly to the above, let us define the resulting graphs as follows: Let $G_1 = G_0 + G_{R,1}$. Let $G_2 = G_0 + G_{R,2}$ Essentially, $k$ controls the frequency of special nodes which shall serve as wired link stations. For $k = \Theta(1)$, the frequency is in line exactly with the bins, and thus the occurence of such wired link stations is 1 in every $\Theta(nr^2)$. For $r = \Theta(r_{con})$ that frequency becomes $\Theta(\frac{1}{\log n})$, and for larger broadcast radius it is sparser:

**Remark 31** *Given $k \geq 1$, the frequency of wired link stations $\frac{|V_R(k)|}{|V_0|}$ is $\Theta(\frac{1}{k^2nr^2})$. Namely, the total number of such stations is $|V_R(k)| = \Theta(\frac{1}{k^2r^2})$.*[6]

Before proceeding to prove results on average path lengths for $G_1 = G_0 + G_{R,i}$, we note that the manner in which $V_R(k)$ is generated can be simulated approximately by simply choosing a total of logarithmically more wired link stations uniformly at randomly from the original set $V_0$. This too follows from Coupon Collection:

**Remark 32** *For any $k$, if every $v \in V_0$ is chosen to be a wired link station with probability $\Theta(\frac{\log(\frac{1}{k^2r^2})}{nk^2r^2})$, then, with high probability, for every $k^2$-bin $\mathcal{B}_{i,j} = \cup_{ki \leq i' \leq (k+1)i, kj \leq j' \leq (k+1)j} B_{i',j'}$ there exists a vertex $v' \in \mathcal{B}_{i,j}$ such that $v'$ is a wired link station. Moreover, all of the vertices in any $k^2$ bin are almost-equiprobable and almost-independent whp.*

Now, we note that the maximum distance of any node in a $k^2$ bin to the corresponding wired link station in the $k^2$ bin is simply bounded by the hop-diameter of the $k^2$ bin:

**Remark 33** *Every node is within $\Theta(k)$ hops of the wired link station in its $k^2$-bin since the $k^2$-bin simply stretches the Manhattan distances of the original 5-lattice-skeleton by $k$.*

This remark shall prove relevant in relating inter-node distances in the graph $G = G_0 + G_R$ to inter-node distances in $G_R$.

## 4 APL and Diameter Bounds

### 4.1 APL and Diameter bounds for $G_1$

Thus, we now proceed concerning inter-node distances for the first model $G_{R,1}$:

Combining Remarks 31, 33, and 214, we obtain our first bounds on the resulting average and worst-case path lengths:

[6] Note that clearly $k$ cannot exceed $\Theta(\frac{1}{r})$.

**Theorem 41** *The diameter and average path lengths for $G_1(k)$ are as follows:* $APL(G_1(k)) = O(Diam(G_1(k))) = O(\min\{2k + \log(\frac{1}{k^2r^2}), \frac{1}{r}\}) = O(\min\{k + \log(n), \frac{1}{r}\})$.

*Proof.* Let $s, t \in V$ be arbitrary. $s$ is within $k$ hops to a node $v \in V_R(k)$, and $t$ is within $k$ hops to a node $w \in V_R(k)$. Moreover, $dist(v, w) = O(\log(|V_R(k)|))$, making the maximum distance at most $2k + \log(|V_R(k)|)$. If $r$ happens to be large, making paths in the original graph (without taking short-cuts in $E_{R,1}$ more convenient, then the maximum distance is the diameter of $G_0$ which is $\Theta(\frac{1}{r})$.

For *any* broadcast radius $r$, it is clear that when $k = \Theta(\frac{1}{r})$, which is the maximum allowable $k$ and corresponds to the situation of placing a constant number $\Theta(1)$ wired-stations, there is no asymptotic improvement in diameters and average path lengths. However, for any other intermediate $k$ and the lowest reasonable broadcast radius $r = \Theta(r_{con})$, the following may be noted:

**Corollary 42** *For $k = O(\log(n))$ and $G_0 = \mathcal{G}(n, \Theta(r_{con}))$, $APL(G_1(k)) = O(Diam(G_1(k))) = O(\log(n))$. On the other hand, for intermediate $k$ such that $k = o(\frac{n}{\log(n)})$ but $k = \omega(\log(n))$, we still obtain asymptotic improvement upon the hop-lengths in $G_0$ as $APL(G_1(k)) = O(Diam(G_1(k))) = O(k) = o(Diam(G_0))$.*

### 4.2 APL and Diameter bounds for $G_2$

The case for the second model $G_2$ is not quite as straightforward as that for the first model due to the fact that the random graph $G_{R,2}(k)$ has a positive probability of being disconnected. So, what can we say? It turns out that a lot can be said the moment that $G_{R,2}(k)$ is superimposed directly upon the nodes of *any* connected graph $G_{init}$:

**Theorem 43** *[Expansion of $G_{connected} + R_{1-out}$][25] Let $G_0$ be any connected graph. Then $G = G_0 + R_{1-out}$, which is the graph formed by adding random 1-out edges to every vertex of $G_0$, is an expander with high probability. In particular, the diameter of $G$is logarithmic in the vertex set.*

Despite the similarities to the situation for $G_1$, there is the technical issue that the vertex set for $G_{R,2}(k)$ is not identical to the vertex set for $G_0$ but rather asymptotically sparser than such. Therefore, we must understand precisely which graph is an expander, and what that yields for the graph $G_2$. In particular, we need to extract a connected graph $G_{base} = (V_{base}, E_{base})$ that is both a relevant function of the original graph $G_0$ and such that there is a one-to-one meaningful correspondence between the vertex set $V_{base}$ and $V_R(k)$ to allow application of Theorem 43.

Visually, if we could just contract the lattice-sub-skeleton formed from the $k^2$-bins into just a $|V_R(k)|$ node lattice, and preserve the meaning of such a contraction for paths in the original graph $G_0$, then we could apply Theorem 43 to our contraction, and then reverse the contraction. We may lose the property

of constant expansion upon reversing our contraction for some $k$, but we will still preserve a bound on path lengths. First, let us state the result, then the proof along the idea above:

**Theorem 44** *The diameter and average path lengths for $G_2(k)$ are as follows:* $APL(G_2(k)) = O(Diam(G_2(k))) = O(\min\{k\log(|V_R(k)|), \frac{1}{r}\}) = O(k\log(\frac{1}{r}))$.

Prior to the theorem proof, let us note the immediate corollary for low broadcast radius, which again notes an exponential improvement in diameter and average path lengths in for poly-logarithmic $k$:

**Corollary 45** *For $k = \Theta(1)$ and $G_0 = \mathcal{G}(n, \Theta(r_{con}))$, $APL(G_2(k)) = O(Diam(G_2(k))) = O(\log(n))$. On the other hand, for intermediate $k$ such that $k = o(\frac{n}{\log^2(n)})$ but $k = \omega(1)$, we still obtain asymptotic improvement upon the hop-lengths in $G_0$ as $APL(G_2(k)) = O(Diam(G_2(k))) = O(k\log n) = o(\frac{n}{\log(n)}) = o(Diam(G_0))$.*

Now to the theorem proof:

*Proof (Proof of Theorem 44).* Consider the graph of node-contractions $G_{contract} = (V_{contract}, E_{contract})$ such that each vertex $v_{i,j} \in V_{contract}$ is precisely the *set* of vertices in $k^2$ bin $\mathcal{B}_{i,j}$ and the edge set $E_{contract}$ is the two-dimensional lattice appropriately defined on $V_{contract}$. We know from Theorem 43 that upon adding random 1-out edges from every $v_{i,j} \in V_{contract}$ we obtain a graph $G' = G_{contract} + R_{1-out}$ with diameter logarithmic in $V_{contract}$. What does this mean?
Consider any two nodes $s, t \in G_0$, say that $s \in \mathcal{B}_{i,j}, t \in \mathcal{B}_{i',j'}$. Let $l =< v_1, v_2, v_3, \ldots, v_{p-1}, v_p >$ be the shortest path between $v_{i,j}$ and $v_{i',j'}$ in $G_{contract}$ (with $v_1 = v_{i,j}, v_p = v_{i',j'}$). First of all, there is an obvious one-to-one correspondence between the vertex sets $V_{contract}$ and $V_R(k)$ which does not change the incidence of the random short-cut edges. The only issue occurs when paths in the contracted graph include a lattice edge, and in that case there is a factor $O(k)$ blow-up in the hop number for the original graph. Thus, for any path $l$ in $G_{contract}$ we may inductively construct the following path $l'$ in $G_{R,2}$ by a sequence of valid sub-path replacements in the place of contracted nodes and the edges between them until no contracted nodes and no contracted edges remain (meaning all nodes are actual vertices of our original graph):

- Base: In accordance with Remark 33, replace $v_1$ in $l$ with the $O(k)$ length shortest-path from $s$ to $v_{i,j}$ in $G_0$. Similarly, replace $v_p$ in $l$ with the $O(k)$ length shortest-path from $v_{i',j'}$ to $t$ in $G_0$.

- Inductive case I: For $1 < t < p-1$, let $v_t$ and $v_{t+1}$ be adjacent nodes in the path $l$ that also remain thus far in our path construction. Uniquely define $v_{x,y} \in v_t, v_{x',y'} \in v_{t+1}$ for $v_{x,y}, v_{x',y'} \in V_R(k)$. Either edge $(v_t, v_{t+1})$ is a lattice edge or a random edge. If it is a random edge, then replace $(v_t, v_{t+1})$ of $l$ with the valid wired edge $(v_{x,y}, v_{x',y'})$. Otherwise, if it is a lattice-edge, then there exists a $O(k)$ length shortest-path $P$ between $v_{x,y}$ and $v_{x',y'}$ in

$G_0$ from Lemma 27 and Remark 33. Thus, in that case, replace $(v_t, v_{t+1})$ with path $P$ between $v_{x,y}$ and $v_{x',y'}$.

– Inductive case II: Let node $v_{x,y}$ be the node following initial wired-station $v_{i,j}$ in the construction thus far. Due to the construction, it must be that $v_{i,j} \in v_1, v_{x,y} \in v_2$. Therefore, similarly to Inductive case I, either there exists a random wired edge between $v_{i,j}$ and $v_{x,y}$, or there is a length $O(k)$ path between the two in $G_0$: Replace accordingly. Operate similarly in making the valid replacement between the node $v_{x',y'}$ preceding final wired station $v_{i'j'}$.

Clearly, the above construction is valid, replacing contracted edges with valid sub-paths at each step. Moreover, at every step, the replacement path is at most a $O(k)$ blow up of a contracted edge. Therefore, the constructed valid path in $G_{R,2}$ is also at most a $O(k)$ multiple of the diameter of $G_{contract}$, which we know by Theorem 43 to be logarithmic in $|V_R(k)|$, finalizing the proof.

## 5 Mixing Time and Connectivity Bounds

### 5.1 Theoretical Bounds on the Mixing Time of $G_1$ and $G_2$

The main results of this section are as follows:

**Theorem 51** *For radius $r = \Theta(r_{con})$, the conductance of both $G_1$ and $G_2$ is $\Omega(\frac{1}{k^2 \log^2 n})$ with high probability.*

From this main theorem this Corollary follows immediately from Remark 1:

**Corollary 52** *For $k = \Theta(polylog(n))$ and radius $r = \Theta(r_{con})$, $G_1$ and $G_2$ are rapidly mixing with high probability.*

Now we prove Theorem 51

*Proof.* First we begin observations that apply to both $G_1$ and $G_2$ and indicate when we must refer specifically to one of the models. From definiton 3 consider the conductance measure $\Phi$ restricted to a given set $X \subset V$, which we will refer to as $\Phi_X$, defined as

$$\Phi_X = \frac{|Cut(X, V \setminus X)|}{d|X|}$$

so that the conductance $\Phi$ can be naturally defined as follows:

$$\Phi = \min_{X \subset V} \Phi_X$$

Recall that in this scenario, $d = \Theta(\log n)$.

In what follows, we refer to the wired link stations as *red* nodes and the only-wireless nodes as *blue* nodes. Now, consider what any set of points $X$ in the plane and the corresponding conductance of such set represents when the underlying

graph is smooth (geo-dense) and geometric: Literally, you may draw boundaries around contiguous point subsets in $X$ such that the cut is proportional to the boundary of your drawing multiplied by the average degree whereas the density of the set is proportional to the area. And, that would remain the case if all points were *blue*. Moreover, in the case that all points are blue, it is also thus well-understood that the set $X$ achieving minimum $\Phi_X$ would correspond to the maximal fat convex contiguous area, as the perimeter to area ratio is thus minimized [14]. But now that we have *red* nodes as well, does this remain true? We argue that it does remain true.

Note first that we may still represent our cut around any set $X$ via drawing boundaries around the contiguous point subsets in $X$, except that now there will be some long "short-cut" edges that we also have to consider arising from the red nodes. Note that the red nodes still remain connected to their close neighbors via the wireless edges, of which there are $\Theta(\log n) = \Theta(d)$ for each red node, whereas there are at most 3 wired edges going out from it (and at most 1 in the case of $G_2$). Therefore, it is clear that, for any set $X$ of strictly less than half the total nodes, if there is a red node $p$ near the border of $X$ but excluded from $X$, the set $Y = X \cup \{p\}$ is such that $\Phi_Y \leq \Phi_X$. In other words, where $\Delta \geq 0$ is the difference between the number of wireless neighbors of $p$ and the number of wired neighbors of $p$, we obtain:

$$\Phi_X = \frac{|Cut(X, V \setminus X)|}{d|X|} > \frac{|Cut(X, V \setminus X)| - \Delta}{d|X + 1|} = \Phi_Y$$

Therefore, it remains true that even in the presence of red nodes we do indeed minimize conductance with contiguous and larger convex point sets. The largest such feasible set is directly any cut that is right down the middle of our unit square. So now that we know what the structure of the set $X$ achieving $\Phi = \Phi_X$ looks like, let us bound $\Phi_X$.

We know from our model and Coupon Collection that the frequency of red nodes is $\Theta(\frac{1}{k^2 \log n})$ with high probability for any large enough contiguous fat convex region (our model is actually stronger than a model required to guarantee this). Any region containing at least $k^2 \log n \log logn$ nodes satisfies this criterion with high probability. So, first let us establish our bounds for such large regions, for both models $G_1$ and $G_2$, and then we consider the expansion of small sets.

First, we make our argument for $G_1$ using the expansion of the red nodes 214. By definition of expansion, w.h.p.

$$\exists c > 0 _\ni |Cut(X, V \setminus X)| \geq \frac{c|X|}{k^2 \log n}$$

This bound follows from the expander linked additions alone and immediately allows us to obtain our bound on the conductance for $G_1$:

$$\Phi = \Phi_X \geq \frac{c}{d_{ave} k^2 \log n} = \frac{c}{k^2 \log^2 n} \tag{4}$$

with high probability.

What happens for $G_2$? Here we must make some use of the facts that the set $X$ achieving the conductance contains half of the total nodes, the red nodes in $X$ are distributed evenly throughout $X$ with frequency $\frac{1}{k^2 \log n}$, and $X$ is chosen independently of the manner in which the red wired-edges are connected. In particular, we wish to use these facts to show that, with high probability, the set of wired edges which cross the cut is of size at least $\Theta(\frac{n}{k^2 polylog(n)})$: For any particular red node $p$, the probability that the wired 1-out neighbor $q$ that $p$ chooses is in $V \setminus X$ is $\frac{1}{2}$. Moreover, over the entire set of red nodes in $X$, this probability remains independently $\frac{1}{2}$ for each node. Thus, this edge selection process may be modeled via Coupon Collection, where the "balls" correspond to red 1-out neighbors chosen by red nodes of $X$, and there are exactly two "bins" which correspond to the sets $X$ and $V \setminus X$ respectively. As the number of balls $\frac{n}{2k^2 \log n}$ is incomparably greater than the number of bins (just 2), the precondition of the Coupon Collection theorem is satisfied, so that we may state this: With very high probability, the number of balls in the first bin is asymptotically the same as the number of balls in the second bin, meaning that both have $\Theta(\frac{n}{4k^2 \log n})$ number of balls. Just considering the meaning of the second bin, this means that the number of wired-neighbors of red-nodes in $X$ that cross into $V \setminus X$ is $\Theta(\frac{n}{4k^2 \log n})$ with high probability. Thus, with high probability,

$$|Cut(X, V \setminus X)| \geq \Theta(\frac{|X|}{k^2 \log n}) \tag{5}$$

Plugging this into the conductance equation, we obtain for $G_2$ that:

$$\Phi_X = \frac{|Cut(X, V \setminus X)|}{d|X|} \geq \Theta(\frac{1}{k^2 \log^2 n}) \tag{6}$$

Finally, we consider what happens under both models for small sets $X$ such that $|X| < k^2 \log n \log\log n$ nodes: To obtain a lower bound on $\Phi_X$ of such situations, it suffices to consider the worst case set conductance on the wireless edges alone (as considering wired links would only improve the bound). And, again, as fat convex contiguous regions minimize set conductance for sets of the same size in a geometric setting, we may directly bound the two-dimensional isoperimetric ratio multiplied by the degree to bound the conductance.

$$\Phi_X \geq \Theta(\frac{d\sqrt{|X|}}{d|X|}) = \Omega(\frac{1}{\sqrt{|X|}}) = \Omega(\frac{1}{k \log n}) \tag{7}$$

In all cases, for both models $G_1$ and $G_2$ and for both large and small sets, we obtain

$$\Phi = \Omega(\frac{1}{k^2 \log^2 n}) \tag{8}$$

This completes our proof.

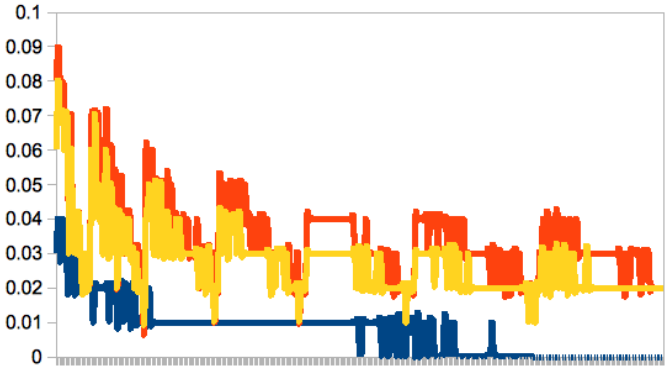


**Fig. 1.** Spectral Gap comparison for $G_1$ (Red), $G_2$ (Yellow), and $\mathcal{G}(n, r)$ (Blue) for network size from 100 to 1620 nodes, radius $r_{con}$, and $k$ parameter of 2

### 5.2 Experimental Bounds on the Algebraic Connectivity

Experiments were conducted for networks of 100 to 1620 nodes. The networks were constructed in a way that is consistent with the models $G_1$ and $G_2$ of the theoretical section, the parameter $k$ was chosen to be 2, and the radius was chosen to be $r_{con}$ exactly, with nodes thrown uniformly at random into the unit square and the edge selections generated in accordance with the described random models. Disconnected $\mathcal{G}(n, r)$ were discarded from consideration. A caveat in our simulations is that we guaranteed a node in the exact center of each bin, because otherwise there were too many discarded geometric graphs due to lack of connectivity. This problem would not be an issue for sufficiently large networks due to the asymptotic theoretical connectivity guarantee, and anyway comparative results are dominated by how edges are chosen rather than precise node locations.

The results can be seen in Figure 1, where the $Y$-axis is the spectral gap of the normalized Laplacian, namely the normalized algebraic connectivity. Notably, the spectral gap for the random geometric graph approaches zero quickly, whereas the spectral gap for $G_1$ and $G_2$ appear to diminish very slowly after 500 nodes. Moreover, note that the number of wired-nodes in comparison to the network size $n$ for $n$ values of 100, 300, 800, 1000, 1300, and 1620 are respectively as follows: 36, 64, 196, 256, 256, 324. The fraction of wired nodes for the network size of 1620 was just $\frac{1}{5}$.

### 5.3 Observations on Experimental Results

Since we are considering the *normalized* Laplacian spectral gap, automatically all results must be between 0 and 1. Note that we considered the normalized spectral gap because we want to avoid the scaling problem that could arise if we compared the non-normalized spectra for graphs with very different degrees.

Further, we consider the Laplacian instead of simply taking the adjacency matrix, because the Laplacian is symmetric, making for faster computations while giving comparable bounds. That all results are less than 0.1 in particular should not be bothersome as well, for two reasons: First, even if we were taking the strong property of being an expander into account, a graph family whose normalized spectral gap never falls below a given constant (e.g. 0.01) would be satisfactory, regardless of the constant. But, we are not attempting to show such a strong property anyway. We are concerned with sufficient expansion, in terms of rapid mixing, which does not even require a constant lower bound, but merely that the *rate* at which the spectral gap falls is slow (in particular, inverse poly-logarithmic).

In fact, as theoretical result Theorem 51 already demonstate that $G_1$ and $G_2$ are indeed rapid mixing, we also notice in the experimental results is that the (yellow) pattern for $G_2$ is extremely similar to the (red) pattern of $G_1$. On the other hand, the spectral gap for the underlying random geometric graph $\mathcal{G}(n, r)$, which is theoretically known to have bad expansion [14], falls to zero far more quickly. As we discarded disconnected cases, it is notable that the spectral gap for $\mathcal{G}(n, r)$ cannot be zero exactly, although it clearly gets arbitrarily close to zero quickly. The spectral gap patterns for $G_1$ and $G_2$, however, fall much more slowly. Thus, the experimental results confirm our theoretical bounds by showing that the spectral gap for $G_1$ and $G_2$ is exponentially larger than that of the underlying random geometric graph.

## 6 Conclusion

We have presented theoretical bounds on the diameter, APL, conductance, and mixing time of sparse random edge additions onto random wireless networks around the connectivity regime, where our bounds are expressed as functions of the wiring frequency. We have also shown experimental results comparing the normalized algebraic connectivities of the underlying random geometric graph to the hybrid models. In particular, we have shown that when the wiring frequency is at least inverse poly-logarithmic, then the subsequent hybrid network exhibits polylogarithmic diameter and mixing time, both of which are exponential improvements to the wireless network about the connectivity regime. We have also shown that there is correspondingly significant asymptotic improvement to the normalized algebraic connectivity which is known to govern network robustness. Taking broadcast radius on the order of the connectivity threshold is particularly important in the case of sensor networks where energy must be preserved and interference diminished, and thus comprises the relevant base graph model as used in much theoretical work on such networks. Nonetheless, the results regarding bounds on diameter and APL are also expressed for general broadcast radii. The mixing time bounds in particular are relevant for distributed gossip applications where it is well established that the performance is dominated by this value. Taken as a whole, this work provides a strong support for hybrid sensor networks.

From a practical standpoint, one may ask how such random wired links should be established atop a wireless sensor network in order that the hybrid model presented be of true relevance. In this regard, we note that the analyses presented is sufficiently general to include an *existing* sparse wired network atop which a wireless network resides. A benefit of theoretical bounds is precisely this lack of restriction of *how* the network details are established. In fact, the random edges of the superimposed links need not even be wired, but may be generated via a sufficiently sharply angled and long ranged directional antenna model as well, as long as the problem of the side lobe may be solved. We point the reader to the existing literature on small worlds for hybrid networks stated in the introduction, as many proposals are given towards the practical aspects of hybrid network creation.

We reiterate that this is the first graph theoretic work to establish solid theoretical foundations of the improvement to graph diameter and mixing time of sparse totally random links (of a non-broadcast nature) upon a random wireless network above and around the connectivity threshold. The two most relevant works with which to compare and contrast results would be those of [10] and [25]. The results of [25] take the base graph to be arbitrary but the wiring probability to so high as to re-wire *every edge* on average. Moreover, that work is concerned with the extremal property of the resulting graph being an expander or not, rather than general expressions of the degree of expansion or mixing time. The arguments used there are beautiful and tight, but are neither sufficiently general to take diminishing random wiring probability, nor sufficiently relevant for the base graph being the random wireless domain in particular. The work of [10], in contrast to [25] does consider asymptotically diminishing wiring probability, and restricts to the relevant model of wireless base graph around connectivity. However, in that case, base station is fixed and a particular type of greedy forwarding is assumed for the routing protocol, so that they actually obtained the *worst* results for the case of totally random links. Moreover, they do not discuss the mixing time at all. Thus, our work may be considered to be a positive complementation to that work in the sense that of the positive results we obtain when routing is both shortest paths based and random.

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